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LXVII. *The Theory of the "Boys" type of Double Resonator* *. *By* E. T. PARIS, *D.Sc.*, *F.Inst.P.*†

§ 1. *Introduction.*

THE "Boys" type of double resonator is used in the construction of sensitive sound-receiving instruments, and consists of a resonator of the "stopped-pipe" variety combined with one of the Helmholtz pattern.

As used for sound-reception the resonator is combined with some form of detector, for example, a Rayleigh disk or a "hot-wire" grid. The detector is mounted in the orifice of the Helmholtz resonator and the magnitude of its response (angular rotation in the case of the Rayleigh disk, and change of electrical resistance in the case of the "hot-wire" grid) provides a measure of the amplitude of the oscillatory air-current in the orifice, and hence of the pressure-amplitude in the original sound-wave.

The function of the double resonator is to magnify the vibrations occurring in the sound-wave so that they may be easily recorded by the detector.

By means of the detector the properties of the double resonator can be investigated experimentally, and, by plotting the response of the detector against the frequency of

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the sound stimulating the resonator, curves are obtained which exhibit characteristic double resonance features. Examples of these curves and methods for calculating the resonance-frequencies from the dimensions of the component resonators have been given in previous papers *.

In this earlier work no allowance was made for any loss of sound-energy from the resonator. Such a loss must, however, always occur owing to the dissipation of sound into heat by the action of viscous forces in the narrow parts of the resonator and to the escape of sound from the mouth of the resonator into the surrounding air.

In the present paper, therefore, an attempt is made to develop a more complete theory of this type of resonator than has yet been given, allowance being made for the loss of sound-energy from both the causes just mentioned. It is clear from the form of the resonator that viscosity losses will be mainly confined to the narrow neck leading into the Helmholtz resonator. The motion in this neck is accordingly supposed to be resisted by a force proportional to the air-current passing through it.

It is convenient to approach the theory of the Boys resonator through that of the simple stopped pipe. A short account of the theory of the pipe is therefore given in § 2, and the theory of the Boys resonator is afterwards developed in § 3.

By way of illustration of the application of the equations obtained some numerical results are given in § 4, and confirmatory experiments are given in § 5.

§ 2. *The Theory of the Stopped-pipe Resonator.*

The behaviour of a stopped pipe under the influence of an external source of sound has been discussed by Helmholtz † and Rayleigh ‡. The difficulty of the investigation lies mainly in the determination of that part of the velocity-potential inside the pipe which is necessary to allow for the escape of sound-energy from the open end, and a satisfactory solution has been obtained only in the case when the diameter of the pipe is small compared with the wave-length of the sound, so that the escaping sound-waves may be assumed to

* Proc. Roy. Soc. A, vol. ci. pp. 393–402 (1922), and Phil. Mag. vol. xlviii. pp. 770–783 (1924).

† Helmholtz's theory is given in 'Vorlesungen über theoretische Physik,' vol. iii. pp. 209–231 (Leipzig, 1898). The investigation originally appeared in Crelle's Journal (1859).

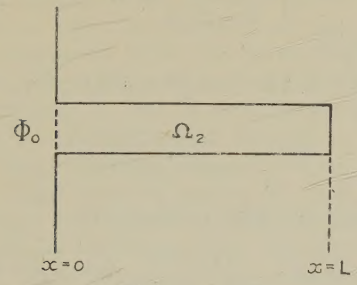
‡ 'Theory of Sound,' vol. ii. (2nd ed.) p. 207.

spread spherically from the centre of the mouth. This condition is sufficiently fulfilled in the case of Boys resonators as ordinarily made, since the diameters of the pipes commonly used are only from 1/10 to 1/20 of a wave-length.

It will be supposed that the mouth of the pipe is provided with an infinite plane reflecting flange, so that the difficulty of making allowance for the scattering of sound from the external surface of the resonator is avoided. So far as the present problem is concerned the principal effect of removing this flange would be to increase the rate at which sound-energy escapes from the open end.

Let the pipe lie with its mouth in the y, z plane (fig. 1) with its axis parallel to the axis of x.

Fig. 1.



A plane wave, $Ae^{ik(at+x)}$, travelling from within the pipe towards the open end, will suffer reflexion at the open end (where part of its energy will be transferred to the outside air) and will give rise to a reflected wave, $\beta Ae^{ik(at-x)}$, where β is in general complex. The amplitude of the reflected wave is $|\beta|$, and if β_1 and β_2 are the real and imaginary parts of β respectively, the change of phase at reflexion is $\tan^{-1}(\beta_2/\beta_1)$. The form of the potential inside the pipe, necessary to allow for the escape of sound from the open end, is therefore

$$\Omega_1 = A(e^{ikx} + \beta e^{-ikx})\, e^{ikat}. \quad . \quad . \quad . \quad (2.1)$$

To Ω_1 may be added any stationary potential Ω_2, of the form $C\cos kx$, which may be necessary to balance the pressures due to external sources in the plane $x=0$. Thus, suppose that the potential Φ_0, at the mouth of the pipe, is due to a plane wave $Fe^{ik(at-x)}$, then $C=2F$ and

$$\Omega_2 = 2F\cos kx \,.\, e^{ikat}. \quad . \quad . \quad . \quad . \quad (2.2)$$

The total potential inside the pipe is then—

$$\begin{aligned}\Omega_{12} &= \Omega_1 + \Omega_2 \\ &= \{A(\epsilon^{\iota kx} + \beta\epsilon^{-\iota kx}) + 2F\cos kx\}\,\epsilon^{\iota kat} \\ &= \{(F+A)\epsilon^{\iota kx} + (F+\beta A)\epsilon^{-\iota kx}\}\,\epsilon^{\iota kat}, \quad . \quad (2.3)\end{aligned}$$

representing two plane waves of amplitudes $|F+A|$ and $|F+\beta A|$ travelling in the directions x-negative and x-positive respectively. The relative amplitudes and phases of the two waves must be adjusted—by a suitable choice of the complex constant A—so that the requirements of the problem are fulfilled in those parts of the pipe away from the mouth. Thus, when the pipe is closed at $x = L$, the following condition must be fulfilled:

$$\left(\frac{\partial\Omega_{12}}{\partial x}\right)_{x=L} = 0. \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

Whence, from (2.3),

$$(F+A)\epsilon^{\iota kL} = (F+\beta A)\epsilon^{-\iota kL}, \quad . \quad . \quad . \quad (2.41)$$

so that

$$A = -\frac{2\iota\sin kL}{\epsilon^{\iota kL} - \beta\epsilon^{-\iota kL}}\,F. \quad . \quad . \quad . \quad (2.42)$$

The potential inside the pipe is therefore

$$\begin{aligned}\Omega_{12} &= 2F\left\{\cos kx - \frac{\iota\sin kL}{\epsilon^{\iota kL} - \beta\epsilon^{-\iota kL}}(\epsilon^{\iota kx} + \beta\epsilon^{-\iota kx})\right\}\epsilon^{\iota kat} \\ &= \frac{2(\beta-1)F}{\beta\epsilon^{-\iota kL} - \epsilon^{\iota kL}}\cos k(x-L)\,.\,\epsilon^{\iota kat}, \quad . \quad . \quad . \quad . \quad (2.5)\end{aligned}$$

indicating, as would be expected, a condition of stationary vibration, the positions of maximum pressure variation alternating with those of maximum velocity at distances of one-quarter of a wave-length.

The quantity denoted by β in the above equation is of frequent occurrence in acoustical problems. In all questions relating to the use of pipes as resonators, allowance must be made for the escape of energy from an open or partially open end, and β enters into the expression for the potential within the pipe. The method of calculating β in a particular case is dealt with below.

Let χ be the potential of the hemispherical sound-waves spreading from the mouth of the pipe, by means of which sound-energy is carried from the pipe into the surrounding medium.

If S_1 be an equipotential surface inside the pipe in the

region of plane waves, taken as near to the mouth as possible, and S_2 be a hemispherical surface drawn outside the pipe with radius r as small as possible, compatible with the condition that the potential χ is the same at all points on S_2, then between S_1 and S_2 the transition will take place from plane waves (Ω_1) inside the pipe to the diverging hemispherical waves (χ) outside the pipe. The problem of finding the relation between Ω_1 and χ, which is essentially that of determining β, has been dealt with by Helmholtz (*loc. cit. supr.*) and Rayleigh *. Their methods, however, differed considerably. Helmholtz assumed the existence of a potential satisfying the general equation of wave-motion in the transition region between S_1 and S_2, and proceeded to find the relation between Ω_1 and χ by several applications of his extended form of Green's theorem. Rayleigh's method, which will be adopted here, is much shorter and proceeds on the assumption that, on account of the smallness of the dimensions (compared with the wave-length) of the region between S_1 and S_2, the motion of the air in this part of the pipe can be treated as that of an incompressible fluid. The conditions to be satisfied in the transition region are then: (*a*) that the total current across S_1 at any instant must be equal to the total current across S_2, and (*b*) that the total current through the transition region must be proportional to the difference between the velocity-potentials at S_1 and S_3.

If the disturbing potential Φ_0 is due to a pure tone of wave-length λ ($=2\pi/k$), the most general form for χ will be

$$\chi = \frac{G}{kr} e^{ik(at-r)}, \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

where r is measured from the centre of the mouth, and G is independent of the space coordinates.

Conditions (*a*) and (*b*) can be expressed by the following equations:

$$\sigma\left(\frac{\partial\Omega_1}{\partial x}\right)_{kx\to 0} = \left(-2\pi r^2 \frac{\partial\chi}{\partial r}\right)_{kr\to 0}, \quad . \quad . \quad . \quad (2.71)$$

$$\sigma\left(\frac{\partial\Omega_1}{\partial x}\right)_{kx\to 0} = c_0\{(\Omega_1)_{kx\to 0} - (\chi)_{kr\to 0}\}, \quad (2.72)$$

where σ is the cross-sectional area of the pipe, and c_0 is the conductance (hydrodynamical) of that part of the pipe lying

* 'Theory of Sound,' vol. ii. pp. 198–201.

between the surfaces S_1 and S_2. By substituting in (2.71) and (2.72) the values of Ω_1 and χ given in (2.1) and (2.6) respectively, we find

$$\iota k\sigma A(1-\beta) = \frac{2\pi G}{k} = c_0 \left\{ A(1+\beta) - \frac{G\epsilon^{-\iota kr}}{kr} \right\}. \quad (2.8)$$

By elimination of G, this gives

$$\frac{1+\beta}{1-\beta} = \iota k\sigma \left(\frac{1}{c_0} + \frac{\epsilon^{-\iota kr}}{2\pi r} \right).$$

Since kr is small compared with unity,

$$\frac{1+\beta}{1-\beta} = \iota k\sigma \left(\frac{1}{c_0} + \frac{1}{2\pi r} - \frac{\iota k}{2\pi} \right). \quad . \quad . \quad (2.81)$$

So that

$$\beta = - \frac{1 - \iota k\sigma \left(\frac{1}{c_0} + \frac{1}{2\pi r} - \frac{\iota k}{2\pi} \right)}{1 + \iota k\sigma \left(\frac{1}{c_0} + \frac{1}{2\pi r} - \frac{\iota k}{2\pi} \right)}. \quad . \quad . \quad (2.9)$$

In this equation r is the radius of the hemispherical equipotential surface S_2. Since the diameter of the mouth is small compared with the wave-length, we may, following Rayleigh *, ignore the term $\frac{1}{2\pi r}$ in comparison with $\frac{1}{c_0}$. In this case, we have as an approximation:

$$\beta = - \frac{1 - \iota k\sigma \left(\frac{1}{c_0} - \frac{\iota k}{2\pi} \right)}{1 + \iota k\sigma \left(\frac{1}{c_0} - \frac{\iota k}{2\pi} \right)}. \quad . \quad . \quad . \quad (2.91)$$

In order to find a numerical value for β in any particular case it is necessary to assign a numerical value to c_0, the conductance of the mouth of the pipe. If the mouth is restricted the calculation of c_0 presents no difficulty, but if it is not restricted there appears to be no method by which a direct calculation of this quantity can be made. An estimate of its value in the case of a pipe of circular cross-section can, however, be made indirectly as follows.

Let the value of β given in (2.91) be inserted in (2.5). Then, after some reduction, the expression for the potential

* 'Theory of Sound,' vol. ii. p. 199.

inside the pipe becomes

$$\Omega_{12} = \frac{2F}{\cos kL - \sigma k\left(\frac{1}{c_0} - \frac{\iota k}{2\pi}\right)\sin kL} \cos k(x-L) \cdot e^{\iota kat}. \quad (2.92)$$

To find the length of the pipe which will show the strongest resonance to a tone of given frequency, Helmholtz put $\tan k\alpha = \sigma k/c_0$, so that

$$\Omega_{12} = \frac{2F}{\frac{\cos k(L+\alpha)}{\cos k\alpha} + \iota\frac{\sigma k^2}{2\pi}\sin kL} \cos k(x-L) \cdot e^{\iota kat}. \quad (2.93)$$

The pressure and velocity variations in the pipe are thus proportional to

$$2F\left\{\frac{\cos^2 k(L+\alpha)}{\cos^2 k\alpha} + \frac{\sigma^2 k^4}{4\pi^2}\sin^2 kL\right\}^{-\frac{1}{2}}. \quad . \quad (2.94)$$

Helmholtz points out that resonance is greatest when $\cos k(L+\alpha)=0$, that is when $(L+\alpha)$ is equal to an odd number of quarter wave-lengths. α is thus the end-correction to the pipe and can either be calculated or determined experimentally. For pipes with diameters small compared with the wave-lengths we may take $\tan k\alpha = k\alpha$, provided there is no restriction of the orifice, and in this case $\alpha = (\sigma/c_0)$ and therefore $c_0 = (\sigma/\alpha)$. If the cross-section of the pipe is circular and of radius R, $\sigma = \pi R^2$ and $\alpha = 0{\cdot}82\,R$, so that $c_0 = (\sigma/\alpha) = 3{\cdot}8\,R$ *.

For a flanged pipe, therefore, (2.91) becomes

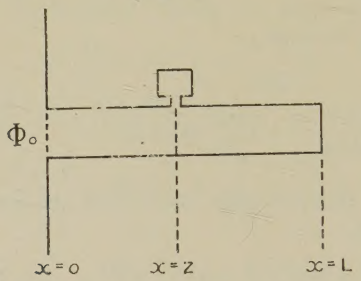
$$\beta = -\frac{1 - \frac{1}{2}(kR^2)^2 - \iota \times 0{\cdot}82\,kR}{1 + \frac{1}{2}(kR^2)^2 + \iota \times 0{\cdot}82\,kR}. \quad . \quad . \quad (2.95)$$

* Another method by which c_0 could be found indirectly from experiment is as follows. Place a source of sound inside the pipe near the closed end. If the motions inside the pipe between the open end and the source are examined (say by means of a Rayleigh disk) there will be found a series of positions of maximum and minimum displacement. These may be regarded as being due to interference between the primary wave from the source and the reflected wave from the open end, and as shown above the ratio of the amplitude of the reflected wave to that of the primary is mod β. The value of mod β is easily calculated from the observed ratio of maximum to minimum displacement, and the value of c_0 could then be discovered which gives the best agreement between the value of mod β found in this way and that calculated from (3.41).

§ 3. *Theory of the Boys Double Resonator.*

In the Boys resonator (fig. 2) let the Helmholtz component be attached to the pipe at a distance l from the open end. In that part of the pipe between $x=0$ and $x=l$ there will

Fig. 2.



be a potential, Ω_{12}, of the form given in (2.3). For brevity, let $A_{12}=F+A$, and $A_{21}=F+\beta A$, so that

$$\Omega_{12} = \{A_{12}\epsilon^{ikx} + A_{21}\epsilon^{-ikx}\}\,\epsilon^{ikat}. \quad . \quad . \quad . \quad (3.1)$$

Let Ω_3 be the potential from $x=l$ to the closed end at $x=L$. Ω_3 will be the potential of stationary waves with a node at $x=L$ and will have the form :

$$\Omega_3 = B\cos k(x-L)\epsilon^{ikat}. \quad . \quad . \quad . \quad . \quad (3.2)$$

Also, since the pressures must be continuous at $x=l$ a necessary condition is $\Omega_{12}=\Omega_3$ when $x=l$, and hence

$$B = \{A_{12}\epsilon^{ikl} + A_{21}\epsilon^{-ikl}\}\,\mathrm{cosec}\,k(l-L). \quad . \quad (3.3)$$

Again, if it be assumed that the disturbance due to the Helmholtz resonator causes the motion in the pipe to depart from the plane form indicated in (3.1) and (3.2) only within a region the dimensions of which are small compared with the wave-length, then the current into the Helmholtz resonator can be equated to the difference between the currents across two equipotential surfaces, normal to the axis of the pipe, one on the positive, and the other on the negative side of $x=l$. This condition can be expressed by

$$\sigma\left\{\frac{\partial\Omega_3}{\partial x} - \frac{\partial\Omega_{12}}{\partial x}\right\}_{x=l} = \frac{dq}{dt},$$

$\frac{dq}{dt}$ being the current (c.c. per sec.) into the Helmholtz resonator.

From (3.2) and (3.3) we have

$$\left(\frac{\partial \Omega_3}{\partial x}\right)_{x=l} = -k\{A_{12}\epsilon^{\iota kl} + A_{21}\epsilon^{-\iota kl}\}\tan k(l-L)\,\epsilon^{\iota kat}. \quad (3.51)$$

and from (3.1)

$$\left(\frac{\partial \Omega_{12}}{\partial x}\right)_{x=l} = \iota k\{A_{12}\epsilon^{\iota kl} - A_{21}\epsilon^{-\iota kl}\}\,\epsilon^{\iota kat}. \quad . \quad (3.52)$$

Let $q = q_0\epsilon^{\iota kat}$, so that $\frac{dq}{dt} = \iota kaq_0\epsilon^{\iota kat}$, then substituting from (3.51) and (3.52) in (3.4) we have

$$\sigma[-k\{A_{12}\epsilon^{\iota kl} + A_{21}\epsilon^{-\iota kl}\}\tan k(l-L) - \iota k\{A_{12}\epsilon^{\iota kl} - A_{21}\epsilon^{-\iota kl}\}] = \iota kaq_0, \quad (3.6)$$

so that

$$q_0 = \frac{\sigma}{a}[\iota\{A_{12}\epsilon^{\iota kl} + A_{21}\epsilon^{-\iota kl}\}\tan k(l-L) - \{A_{12}\epsilon^{\iota kl} - A_{21}\epsilon^{-\iota kl}\}]$$

$$= -\frac{\sigma}{a}\operatorname{cosec} k(l-L)[A_{12}\epsilon^{\iota kl} - A_{21}\epsilon^{-\iota kl}]. \quad . \quad . \quad (3.61)$$

A second expression for q_0 can be obtained by considering the equation of motion of the air in the neck of the Helmholtz resonator. Since the pressure variation is $\rho\frac{d}{dt}(\Omega_{12})_{x=l}$, ρ being the density of the air, this equation is *

$$\frac{d^2q}{dt^2} + 2h\frac{dq}{dt} + \omega_0^2 q = c\frac{d}{dt}(\Omega_{12})_{x=l}, \quad . \quad . \quad (3.7)$$

where h is the damping factor of the Helmholtz resonator, c is the hydrodynamical conductance of the neck, and $\omega_0^2 = a^2c/Q$, Q being the internal volume of the resonator and a the velocity of sound. ω_0 is equal to 2π times the resonance frequency of the Helmholtz component. Thus, writing ω for ka, we have

$$q_0 = \frac{\iota\omega c}{(\omega_0^2 - \omega^2) + 2\iota\omega h}\{A_{12}\epsilon^{\iota kl} + A_{21}\epsilon^{-\iota kl}\}. \quad . \quad (3.71)$$

Or, if Δ is written for $\omega_0(\omega_0/\omega - \omega/\omega_0)$,

$$q_0 = \frac{c}{2h - \iota\Delta}\{A_{12}\epsilon^{\iota kl} + A_{21}e^{-\iota kl}\}. \quad . \quad . \quad (3.72)$$

* Rayleigh, 'Theory of Sound,' vol. ii. p. 198.

Equating the values of q_0 in (3.61) and (3.72), we obtain

$$A_{12}\epsilon^{\iota kL} - A_{21}\epsilon^{-\iota kL}$$

$$= -\frac{ac}{\sigma(2h-\iota\Delta)}\cos k(l-L)\{A_{12}\epsilon^{\iota kl} + A_{21}\epsilon^{-\iota kl}\}, \quad (3.8)$$

so that

$$A_{21} = A_1{}^2 \frac{\epsilon^{\iota kL} + \frac{ac}{\sigma(2h-\iota\Delta)}\cos k(l-L)\,.\,\epsilon^{\iota kl}}{\epsilon^{-\iota kL} - \frac{ac}{\sigma(2h-\iota\Delta)}\cos h(l-L)\,.\,\epsilon^{-\iota kl}}$$

$$= A_{12}\epsilon^{2\iota kL}\frac{1+\frac{ac}{\sigma(2h-\iota\Delta)}\cos k(l-L)\,.\,\epsilon^{\iota k(l-L)}}{1-\frac{ac}{\sigma(2h-\iota\Delta)}\cos k(l-L)\,.\,\epsilon^{-\iota k(l-L)}}. \quad (3.81)$$

Let

Then
$$\mu = (A_{21}/A_{12})\epsilon^{-2\iota kL}. \quad . \quad . \quad . \quad . \quad (3.82)$$

Hence
$$\frac{A_{21}}{A_{12}} = \mu\epsilon^{2\iota kL} = \frac{F+\beta A}{F+A}. \quad . \quad . \quad . \quad (3.83)$$

Thus
$$A = -\frac{1-\mu\epsilon^{2\iota kL}}{\beta-\mu\epsilon^{2\iota kL}}F. \quad . \quad . \quad . \quad . \quad (3.84)$$

$$q_0 = \frac{cA_{12}}{2h-\iota\Delta}(\epsilon^{\iota kl} + \mu\epsilon^{2\iota kL}\epsilon^{-\iota kl})$$

$$= \frac{c}{2h-\iota\Delta}\frac{(\beta-1)F}{\beta-\mu\epsilon^{2\iota kL}}(\epsilon^{\iota kl} + \mu\epsilon^{2\iota kL}\epsilon^{-\iota kl})$$

$$= \frac{c}{2h-\iota\Delta}\frac{(\beta-1)F}{\beta\epsilon^{-\iota kL} - \mu\epsilon^{\iota kL}}\{\epsilon^{\iota k(l-L)} + \mu\epsilon^{-\iota k(l-L)}\}. \quad (3.9)$$

The theory of the Helmholtz resonator shows that the value of q_0 corresponding to a primary potential $\phi_0\epsilon^{\iota kat}$ at the mouth of the resonator is

$$q_0 = \frac{c}{2h-\iota\Delta}\phi_0\,; \quad . \quad . \quad . \quad . \quad (3.91)$$

so that (3.9) indicates that, in the case of the Boys resonator, the current in the neck of the Helmholtz component is the same as would be induced, in an unenclosed space, by a primary potential

$$\frac{(\beta-1)F}{\beta\epsilon^{-\iota kl} - \mu\epsilon^{\iota kL}}\{\epsilon^{\iota k(l-L)} + \mu\epsilon^{-\iota k(l-L)} \quad . \quad . \quad . \quad (3.92)$$

The theory of the simple stopped-pipe resonator, as given in § 2, shows that the potential at a distance from the open end is (*cf.* equation (2.5)):

$$\frac{(\beta-1)\mathrm{F}}{\beta\epsilon^{-ik\mathrm{L}}-\epsilon^{ik\mathrm{L}}}\{\epsilon^{ik(l-\mathrm{L})}+\epsilon^{-ik(l-\mathrm{L})}\}. \quad . \quad . \quad (3.93)$$

Hence, comparing (3.92) and (3.93), we see that the factor μ occurring in both numerator and denominator of (3.92) represents the reaction of the Helmholtz resonator on the potential within the pipe. As would be expected, the reaction depends not only on the distance of the resonator from the closed end but also upon its damping, tuning, the conductance of the orifice, and the cross-sectional area of the pipe.

Equation (3.9) completes the solution of the problem so far as it is required for present purposes, for it enables us to calculate the amplitude of the air displacements in the neck of the Helmholtz resonator for any combination of pipe and resonator within the limits imposed by the assumption as to the smallness of the diameter of the pipe relative to the wave-length.

For the purpose of numerical calculation it is convenient to make use of the conception of what may be called the "magnification" of the resonator.

As stated previously, the function of the double resonator is to magnify the vibrations occurring in the incident sound-wave so that they may be easily recorded by the detector, the latter being placed in the narrow neck of the Helmholtz component. A proper measure of the magnification due to the resonator is therefore the ratio of the maximum velocity (or displacement) occurring in the neck of the Helmholtz resonator to the maximum velocity (or displacement) occurring in the incident sound-wave *.

If A is the cross-sectional area of the neck of the Helmholtz component, the maximum velocity occurring in the neck is $\frac{1}{\mathrm{A}}\left|\frac{dq}{dt}\right|$. The maximum velocity in the incident plane wave $\mathrm{F}\epsilon^{ik(at-x)}$ is $|ik\mathrm{F}|$, but owing to reflexion the pressure variation due to the incident wave at the mouth of the resonator is double what it would be if there were no flange. The question arises as to whether the (supposed infinite) flange is or is not to be regarded as part of the resonator. The best way out of this difficulty is probably

* *Cf.* 'Science Progress,' vol. xx. p. 69 (1925).

to adopt as the standard of comparison that velocity which in a plain progressive wave would be associated with a pressure-variation equal to that due to the primary wave at the mouth of the resonator—in this case $|2\iota k\mathrm{F}|$. Thus, if M denote the magnification :

$$\mathrm{M} = \left(\frac{1}{\mathrm{A}}\left|\frac{dq}{dt}\right|\right) \Big/ |2\iota k\mathrm{F}|. \quad . \quad . \quad . \quad (3.94)$$

Whence, from (3.9),

$$\mathrm{M} = \frac{1}{2}\frac{a}{\mathrm{A}}\left|\frac{c}{2h-\iota\Delta}\,\frac{\beta-1}{\beta\epsilon^{-\iota k\mathrm{L}}-\mu\epsilon^{\iota k\mathrm{L}}}\{\epsilon^{\iota k(l-\mathrm{L})}+\mu\epsilon^{-\iota k(l-\mathrm{L})}\}\right|. \quad (3.95)$$

Some numerical values of M, showing the variation with frequency of the incident sound, are given in the next section.

§ 4. *Numerical Results.*

In this section a short account is given of the calculation of M by means of (3.95) in the case of a Boys resonator in which both components (Helmholtz resonator and stopped pipe) have a fundamental frequency (n_0) of 256 vibrn./sec.

The dimensions and other characteristics of the Helmholtz resonator were taken to be such as are normally employed for the construction of hot-wire microphones, the actual values selected being the following :—

Diameter of orifice $= d = 0{\cdot}5$ cm.
Conductance of orifice $= c = 0{\cdot}125$ cm.
Damping factor at 256 vibrn./sec. $= h = 50$ sec.$^{-1}$.

The magnification associated with the Helmholtz resonator when used alone appears as a factor in the expression for M given in (3.95), for it is easily shown that it is equal to

$$\frac{a}{\mathrm{A}}\left|\frac{c}{2h-c\Delta}\right|.$$

So that, if this is denoted by M_1, (3.95) can be written :

$$\mathrm{M} = \mathrm{M}_1 \times \frac{1}{2}\left|\frac{\beta-1}{\beta\epsilon^{-\iota k\mathrm{L}}-\mu\epsilon^{\iota k\mathrm{L}}}\{\epsilon^{\iota k(l-\mathrm{L})}+\mu\epsilon^{-\iota k(l-\mathrm{L})}\}\right|. \quad (4.1)$$

Values of M_1 corresponding to values of the interval n/n_0 between 0·80 and 1·20 are given in Table I. The velocity of sound (a) has been taken to be 33760 cm./sec.

In calculating M_1 it has been assumed that the damping

factor h remains constant throughout the range of frequencies dealt with. In practice it is found that h varies with frequency, but the information at present available is insufficient to allow this fact to be taken into consideration.

The magnification associated with the stopped pipe is seen from (2.5) to be

$$M_2 = \left| \frac{\beta - 1}{\beta \epsilon^{-ikL} - \mu \epsilon^{ikL}} \right| . \quad . \quad . \quad . \quad . \quad (4.2)$$

M_2 can also be introduced as a factor in the expression for M, but for numerical calculation it is generally better to retain M in the form given in (4.1). It is of interest, however, to compare the magnification belonging to the pipe-component with that obtainable from the double resonator, and for this reason some values of M_2 have been included in Table I.

The reduced length of the pipe is equal to a quarter of the fundamental wave-length, that is, to $a/4n_0$. Hence, if R is the radius of the cross-section,

$$L + 0{\cdot}82\,R = a/4n_0 . \quad . \quad . \quad . \quad . \quad (4.3)$$

In order to define the shape of the pipe (that is, the ratio of length to diameter), let $L = mR$. We then have from (4.3)

$$\left.\begin{aligned} R &= \frac{a}{4n_0} \frac{1}{m + 0{\cdot}82} \\ L &= \frac{a}{4n_0} \frac{m}{m + 0{\cdot}82} \end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad (4.4)$$

Hence

$$kR = \frac{\pi}{2} \frac{n}{n_0} \frac{1}{m + 0{\cdot}82} , \quad . \quad . \quad . \quad . \quad (4.51)$$

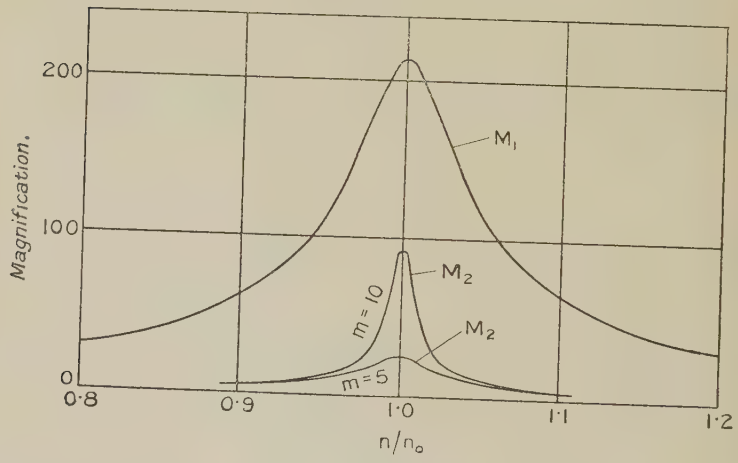
$$kL = \frac{\pi}{2} \frac{n}{n_0} \frac{m}{m + 0{\cdot}82} . \quad . \quad . \quad . \quad . \quad (4.52)$$

To show the effect of changing the cross-sectional area of the pipe, the two cases $m = 5$ and $m = 10$ were selected for calculation. Values of β were first calculated by means of (2.95) from the values of kR given by (4.51). Corresponding values of kL were then found from (4.52) and finally M_2 computed by means of (4.2). The values of M_2 so found are given in Table I.

TABLE I.

n/n_0.	M_1.	M_2.		n/n_0.	M_1.	M_2.	
		$m=5$.	$m=10$.			$m=5$.	$m=10$.
0·80	29	—	—	1·02	181	21	—
2	33	—	—	4	133	14	—
4	38	—	—	6	101	10	11
6	43	—	—	8	80	8	8
8	51	—	—	10	67	6	6
90	61	6	6	2	57	—	—
2	75	7	8	4	49	—	—
4	96	10	10	6	44	—	—
6	130	13	—	8	39	—	—
8	180	20	30	1·20	36	—	—
1·00	215	28	95				

Fig. 3.



In fig. 3, M_1 and M_2 are plotted against n/n_0. It will be seen that the magnification due to the Helmholtz resonator is much greater than that obtainable from the stopped pipe. The increased sharpness of resonance shown by M_2 when m is changed from 5 to 10 is very marked, the reason being that sound-energy escapes less readily from the narrower pipe.

The calculation of M for the double resonator by means of (4.1) presents no difficulty. A preliminary calculation of μ is necessary, and this can be effected by means of the formula

$$\mu = \frac{1+\dfrac{ac}{\sigma(2h-\iota\Delta)}\cos k(l-\mathrm{L})\,.\,\epsilon^{\iota k(l-\mathrm{L})}}{1-\dfrac{ac}{\sigma(2h-\iota\Delta)}\cos k(l-\mathrm{L})\,.\,\epsilon^{-\iota k(l-\mathrm{L})}} \quad . \quad . \quad (4.6)$$

σ is the cross-sectional area of the pipe, so that

$$\sigma = \pi \mathrm{R}^2 = \pi\left(\frac{a}{4n_0}\frac{1}{m+0{\cdot}82}\right)^2 . \quad . \quad . \quad (4.7)$$

The following three cases were selected for the calculation of M :—

Case I. $m=5$, $l=\mathrm{L}$.—The Helmholtz resonator is here at the closed end of the pipe.

Case II. $m=10$, $l=\mathrm{L}$.—The Helmholtz resonator is again at the closed end of the pipe, but the diameter of the latter is only about half its vale in Case I.

Case III. $m=10$, $l=0{\cdot}3\,\mathrm{L}$.—The diameter of the pipe is the same as in Case II, but the Helmholtz resonator is inserted through the wall of the pipe at a distance of $0{\cdot}3\,\mathrm{L}$ from the open end.

The numerical values of M calculated for these three cases are given in Table II.

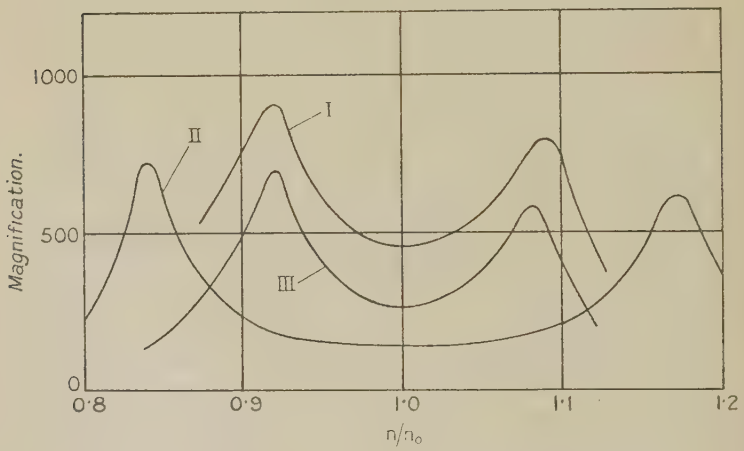
TABLE II.

n/n_0.	I. $m=5$.	II. $m=10$.	III. $m=10$.	n/n_0.	I. $m=5$.	II. $m=10$.	III. $m=10$.
0·80	—	230	—	1·02	476	—	281
2	—	416	—	4	534	—	—
4	—	733	137	6	635	171	426
6	—	487	210	8	773	186	587
8	598	309	326	1·10	743	217	436
90	770	231	504	2	450	270	229
2	910	191	698	4	—	369	—
4	699	169	461	6	—	545	—
6	554	—	—	8	—	578	—
8	485	149	292	1·20	—	344	—
1·00	463	146	274				

These values of M are plotted against n/n_0 in fig. 4, in which the vertical scale is only one-fifth of that used in fig. 3. The following points may be noted in connexion with these curves:—

(i.) The curve I ($m=5$, $l=\mathrm{L}$) shows the greatly increased magnification obtained by the use of double resonance. It will be seen from Table I, or fig. 3, that the greatest magnifications obtainable from the Helmholtz resonator or stopped pipe alone are 215 and 28 respectively. When the two are combined, however, the magnification does not fall below 450 between $n/n_0=0{\cdot}86$ to $n/n_0=1{\cdot}12$, that is from 220 to 287 vibrn./sec.; while at $n/n_0=0{\cdot}92$ and $n/n_0=1{\cdot}085$ it rises to 910 and 810 respectively.

Fig. 4.



(ii.) The curve II ($m=10$, $l=\mathrm{L}$) shows the effect of reducing the cross-sectional area of the pipe. Although the magnification obtainable from the pipe alone is greater than in Case I, namely, 95 as compared with 28, the reduction in cross-section has been accompanied by a tightening of the coupling between the two components, with the result that the two resonance frequencies of the double resonator are further apart, and the general level of the magnification is lowered. The small variation in magnification in the region between about $n/n_0=0{\cdot}96$ and $1{\cdot}04$ is noticeable. Instruments made with double resonators of this type are useful in cases in which it is required to record the strength of a sound the pitch of which is liable to undergo small variations.

(iii.) The resonance frequencies as found from the curve in fig. 4 show a satisfactory agreement with those calculated from the simple formula previously obtained by neglecting the loss of sound-energy from the resonator. Thus, curve I shows that resonance frequencies occur at approximately $n/n_0=0{\cdot}92$ and $1{\cdot}09$, while the formula previously given *, which may be written

$$\tan\frac{\pi}{2}\frac{n}{n_0}=\frac{\sigma}{ac}\Delta, \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

indicates that resonance frequencies occur when $n/n_0=0{\cdot}91$ and $1{\cdot}09$. Similarly, curve II indicates resonance at $n/n_0=0{\cdot}84$ and $1{\cdot}17$, these values being identical with those found from a graphical solution of (4·8).

(iv.) The curve III shows the effect of moving the Helmholtz resonator towards the open end of the pipe. The resonance peaks are brought nearer together again, and occupy nearly the same positions as when $m=5$. The general level of the magnification is raised, but is maintained for a smaller range of frequencies. Previous experiments have indicated that enhanced sensitiveness may sometimes be obtained by a suitable choice of the distance of the Helmholtz resonator from the open end of the pipe. The conditions under which this may occur should be derivable from the equations contained in the present paper, but a full investigation has not yet been carried out.

§ 5. *Experimental Results.*

Some observations will now be described which were undertaken with the object of providing an experimental check on the theory developed in the preceding sections.

No means being available for measuring the actual value of the magnification of resonators, attention was directed to obtaining resonance curves the forms of which could be compared with the theoretical curves shown in fig. 4. For this purpose a singly resonated hot-wire microphone was chosen with a natural frequency of 256 vibrn./sec. The characteristics of the Helmholtz resonator were very nearly the same as those used in § 4, their actual values being $c=0{\cdot}115$ cm., $h=49$ sec.$^{-1}$.

For combining with this Helmholtz resonator, two ebonite stopped-pipes were made whose dimensions were (*a*) L= 29 cm., 2R=12·1 cm; (*b*) L=31 cm., 2R=5·7 cm. Thus

* Equation (12), Proc. Roy. Soc. A, ci. p. 395 (1922).

for the pipe (*a*) we have $m=4{\cdot}8$, and for the pipe (*b*), $m=10{\cdot}9$.

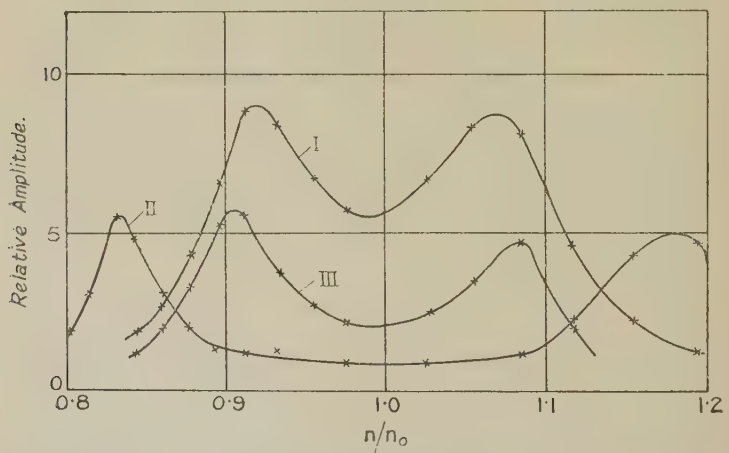
To correspond with the three cases dealt with in § 4, the following arrangements of Helmholtz resonator and pipe were made :—

I. The orifice of the Helmholtz resonator was inserted through the closed end of the wider pipe. Thus $m=4{\cdot}8$, $l=\mathrm{L}$.

II. The orifice of the Helmholtz resonator was inserted through the closed end of the narrower pipe, so that $m=10{\cdot}9$, $l=\mathrm{L}$.

III. The orifice of the Helmholtz resonator was inserted through the wall of the narrower pipe at a distance $\frac{3}{10}$ L from the open end so that $m=10{\cdot}9$, $l=\frac{3}{10}$ L.

Fig. 5.



The experimental resonance curves for these three cases are shown in fig. 5, the crosses indicating the observed points. The method by which they were obtained was substantially that described in previous papers. The source of sound was provided by a specially designed thermionic valve oscillator, the current from which actuated the diaphragm of a "loud-speaker." To avoid inequalities in acoustic output due to resonances, the horn of the loud-speaker was removed.

The observations were made in the open, the resonators to be examined being laid on the ground at a distance of 0·8 metre from the source of sound. The surface of the

ground thus represented the infinite flange assumed in the theory.

The bridge method of observing the response of the hot-wire grid was employed, the ohmic changes being proportional to the square root of the deflexions shown on a microammeter. The scale of relative amplitudes employed in fig. 5 was thus arrived at by taking the square root of the observed deflexions. The hot-wire grids employed were of standard type with hot resistance about 330 ohms and maximum safe working current of 27 milliamperes. A relative amplitude of 10 in fig. 5 corresponds to a resistance change of about 1 ohm.

A comparison of fig. 5 with fig. 4 shows that, having regard to the small disparities between the dimensions and other characteristics of the resonators used in practice with those assumed in the theory, the agreement between the observed and calculated curves is satisfactory and indicates that the theory as developed in § 4 is substantially correct.

Improvement could probably be effected by obtaining a more accurate value for β and by taking into account the variation of h with frequency.

§ 6. *Summary.*

A theory of the "Boys" type of double resonator is developed in which allowance is made for the loss of sound-energy from the resonator owing to (i.) the dissipation of sound into heat by the operation of viscous forces in the narrow neck leading into the Helmholtz resonator, and (ii.) the escape of sound into the surrounding air through the open end of the pipe component. Some numerical results are given which show the changes indicated by theory in the form of the resonance curves caused by (*a*) diminishing the cross-sectional area of the pipe, and (*b*) moving the Helmholtz resonator from the closed end towards the open end of the pipe. The forms of the calculated resonance curves are found to be in fair agreement with observations. The resonance frequencies as found from the more complete theory now given agree well with those found from the approximate theory previously developed.

Air Defence Experimental Establishment,
Woolwich.
May, 1926.

LXVIII. *Regularities in the Spectrum of Ionized Silver.* *By* C. S. Beals, *M.A.*, *D.I.C.**

[Plate XIV.]

Introductory.

THE arc spectra of copper, silver, and gold, elements occupying the sub-group in the first column of the periodic table, have been shown to be similar in that they all three contain doublet series. According to the view recently advanced by Sommerfeld, however, the arc spectrum of silver would be expected to be of a more simple character than the corresponding spectra of copper and gold. It was formerly believed that the character of the terms in a spectrum was entirely determined by the azimuthal quantum numbers associated with successive orbits occupied by the outermost electron of the atom. According to more recent theories of atomic structure as advanced by Russell and Saunders †, Heisenberg ‡, Hund §, and others, the character of a spectral term is determined by what may be called a group quantum number which depends on the resultant of the moments of momenta of all the electrons external to the rare-gas-like core of the atom. The simple character of the spectra of the alkali metals has accordingly been attributed to the fact that they each have but a single electron external to the rare-gas-like core. The prediction which was made by Sommerfeld that the spectrum of silver would be similar to those of the alkali metals in containing only doublet terms, is based on the belief that the atomic configuration which obtains for palladium (the element preceding silver in the periodic table) partakes somewhat of the rare-gas-like character. The atomic structure of palladium consists of a completed shell of 10 4_3 electrons external to the rare-gas-like (Krypton) core. This configuration of electrons apparently possesses an inherent stability somewhat similar to that which obtains for the atoms of the rare-gases. As a consequence of this, when a single electron is added to form the silver atom the nature of the terms in the spectrum should depend solely on the successive orbits

* Communicated by Prof. A. Fowler, F.R.S.

† Russell & Saunders, Astrophys. Journ. vol. lxi. p. 38 (1925).

‡ Heisenberg, *Zeit. f. Phys.* vol. xxxii. p. 841 (1925).

§ Hund, *Zeit. f. Phys.* vol. xxxiii. p. 335 (1925).

occupied by this single outer electron. The spectrum would thus be expected to be alkali-like in character, and to contain only doublet terms.

The spectrum of copper is less simply related to those of the alkalies. The arrangement of doublet series terms in the spectrum of copper would appear to indicate that the atom of copper in its normal state is exactly analogous with that of silver and consists of a shell of 10 3_3 electrons and a single 4_1 electron external to the rare-gas-like (argon) core. The distinction between the two is believed to be found in the smaller stability of the shell of 10 3_3 electrons in the atom of copper. This is evidenced by the fact (believed to be established by spectroscopic data) that of the 10 electrons external to the argon core in the neutral atom of nickel, only 8 occupy 3_3 orbits, the remaining two being found in 4_1 orbits, From this it may be inferred that terms in the spectrum of copper may not only be due to successive orbits occupied by the valence electron, but may also be affected by the displacement of additional electrons from 3_3 orbits to outer orbits when the atom is in an excited condition. These considerations have led Hund * to predict the appearance of quartet terms in the spectrum of copper. As the atomic configuration for gold is believed to be closely analogous with that of copper, it seems reasonable to extend Hund's prediction to include the spectrum of gold.

In so far as experimental data are at present available, they are in agreement with the theoretical predictions contained in the preceding paragraph. The spectrum of copper † has already been shown to contain quartet terms in addition to its doublet series system, while evidence is not lacking that a quartet system will presently be shown to be a feature of the gold spectrum ‡.

The arc spectrum of silver contains a smaller number of lines than the corresponding spectra of copper and gold, and a majority of these lines have been shown to belong to a doublet system. The unclassified lines, which are found mainly in the ultra-violet, form a very puzzling feature of the silver arc spectrum. There is no evidence at present available which would connect them with a quartet system, while at the same time it seems fairly certain that they do not form part of the regular doublet

* Hund, *Zeit. f. Phys.* vol. xxxiii. p. 335 (1925).
† Beals, Proc. Roy. Soc. A, vol. cxi. (May 1926).
‡ McLennan & McLay, Proc. Roy. Soc. A, vol. cviii. p. 515 (1925).

series. Many of these lines show curious fluctuations in intensity. For example, the lines λ 2309·63 and λ 2374·90 are very diffuse and so strong as to dominate the spectrum in this region under ordinary arc conditions. When the arc is passed *in vacuo* they disappear completely. It seems possible that some of these lines may form an anomalous feature of the silver spectrum, which has no counterpart among previously recognized spectral structures.

From what has already been said concerning the structure of the silver atom, it may be deduced that the removal of the outer 5_1 electron would leave a configuration closely resembling that found in the atom of neutral palladium; *i.e.*, a completed shell of 10 4_3 electrons external to the rare-gas-like core of the atom. The spectrum of the singly-ionized atom of silver would accordingly be expected to resemble in its general features the arc spectrum of palladium. Fortunately, the necessary data concerning the palladium spectrum are now available, analyses of this spectrum having been published by the present writer *, by Catalan †, and by McLennan and Smith ‡. Most of the strong palladium arc lines have been included in a singlet and triplet system. The frequencies of these lines have been shown to be due to transitions from a number of levels defined by the terms ^3P, $^3D'$, ^3F, ^1P, $^1D'$, ^1F to a number of lower levels corresponding to the terms ^1S, ^3D, and ^1D. The ^1S term has been found to be the fundamental term, a fact that had been predicted on theoretical grounds by Hund and Catalan. Since the atomic configurations for Pd I and Ag II are believed to be precisely analogous, a ^1S term should also appear as the fundamental term of the Ag II spectrum. The other terms of the palladium spectrum would also be expected, but it is not certain that they would appear in the same order.

The present paper contains the results of an investigation of the first spark spectrum of silver. The work has included the separation of Ag II lines from arc lines and lines corresponding to higher ionizations, a study of Zeeman effects, and a discussion of the numerical relations which have been found to exist in the Ag II spectrum.

* Beals, Proc. Roy. Soc. A, vol. cix. p. 369 (1925).

† Catalan, *Zeit. f. Phys.* vol. xxxv. p. 449 (1926).

‡ McLennan & Smith, Trans. Roy. Soc. Canada, vol. xx. p. 1 (Jan. 1926).

The Separation of Arc and Spark Lines.

In order to sort out the enhanced lines and to assign them to a particular stage of ionization, the spectrum of silver was observed under varying intensities of excitation. The most useful sources of light for this purpose were an ordinary arc in air and a strongly-condensed spark obtained with a large resonance transformer. Intermediate stages between the arc and spark could be obtained by employing varying amounts of inductance in the spark circuit. Some difficulty was experienced due to the masking of lines by band spectra, and this was remedied by passing the arc and spark in hydrogen.

The behaviour of arc and spark lines is illustrated in Pl. XIV., in which arc and spark photographs have been compared. The lines corresponding to the first ionization (Ag II) are fairly strong in the arc and might easily be mistaken for arc lines. In the spark spectrum the Ag II lines are strongly enhanced, while the arc lines become weaker. The spark photographs which were obtained without the use of inductance in the spark circuit show a large number of lines which do not appear or are extremely feeble in the arc. It appears probable that these lines may belong to the second or higher ionization.

Those lines which are considered to belong to the spectrum of singly ionized silver have been included in Table I. In this table is indicated the wave-length and wave-number of each line and an estimate of its relative intensity. Wherever possible Eder's wave-lengths have been used, and where they are not available the values of Exner and Haschek and Huppers have been given. The region of the spectrum which has been examined extends from λ 6500–1850, but no lines of wave-length longer than λ 3000 have definitely been assigned to the spectrum of Ag II.

Observations of Zeeman Effects.

Unfortunately, many of the shorter wave-length lines of Ag II lie in a region of the spectrum which is not accessible to ordinary Zeeman Effect observations. It has not been found possible to use the higher orders of the grating for observation of lines below λ 2240, and the wave-length separation of components in this region of the spectrum is so small that first- and second-order observations are of little use. As a consequence of this the Zeeman measurements are limited to those lines which occur in the region λ 2240–2940.

TABLE I.

λ.	(Int.).	ν.	Zeeman Effect.	Combination.
2929·37	(3)	34127·1	0·90	D–L
2767·54	(4)	36122·5	(0·00)0·97	D–N
2660·49	(3)	37575·9	.....	D–O
2535·30	(4)	39431·3	0·81	C–L
2506·63	(4)	39882·3	0·84	C–M
2473·84	(4)	40410·7	(0·00)1·53	D–P
2447·93	(5)	40838·5	(0·28)1·00	
2437·81	(10)	41008·0	(0·00)1·18	B–L
2413·21	(9)	41426·1	(0·00)1·04	C–N
2411·37	(5)	41457·6	(0·00)1·23	B–M
2357·92	(4)	42397·3	(0·00)0·44	
2331·36	(5)	42880·2	(0·00)0·99	C–O
2324·65	(6)	45002·9	0·76	B–N
2320·23	(5)	43086·0	(0·00)1·18	D–Q
2317·03	(5)	43145·5	(0·00)1·30	
2279·96	(5)	43846·8	(0·00)1·04	D–R
2248·72	(4)	44455·9	0·83	B–O
2246·37	(6)	44502·4	(0·00)1·19	
2229·49	(6)	44839·4		
2186·73	(6)	45716·0	.....	C–P
2166·42	(6)	46144·6	.....	A–L
2145·55	(7)	46593·3	.....	A–M
2120·36	(6)	47146·9		
2113·76	(7)	47294·1	.....	B–P
2065·82	(8)	48390·8	.....	C–Q
2033·84	(3)	49152·1	.....	C–R
2015·72	(5)	49594·0		
2000·56	(6)	49969·8	.....	B–Q
1993·55	(6)	50145·4	.....	C–S
1932·76	(4)	51722·3	.....	B–S

The experimental arrangements for the observation of Zeeman effects has already been described *, and has been used without sensible alteration in the present work. An arc in air between silver poles served as a source of light. The Ag II lines appear in the spark with considerably greater intensity than in the arc, but they are less clearly defined and a number of them show definite reversals. For

* Beals, Proc. Roy. Soc. A, vol. cix. p. 369 (1925).

this reason sole reliance has been placed in the arc, and little difficulty has been experienced in obtaining the lines with sufficient intensity. The photographs have been taken in the third, fourth, and fifth orders of a 10-foot concave grating. Times of exposure have varied from 30 minutes to 2 hours.

The measurements of Zeeman components have been included in Table I. In accordance with the usual practice, the separations have been expressed in terms of the normal Zeeman effect as shown by the separation of components in lines of a singlet system. The value of the normal Zeeman effect has been derived from the two calcium lines λ 3933·66 and λ 3968·46 which appeared as impurities in the silver spectrum. The resolutions shown by these line are as follows :—λ 3933·66, (·33) 1·00, 1·66, λ 3968·46, (·66) 1·33.

Structure of the Spectrum.

As a first step in the analysis of the Ag II spectrum a search was made for any constant wave-number differences which might exist among the lines. A number of such relations have been discovered, and the results of this part of the investigation have been incorporated in Table II. As

Table II.

	A.	B.	C.	D.
	100,000·0	94,863·9	93,287·1	87982·2
		(5136·1)	(1576·8)	(5304·9)
L 53855·5.....	46144·6(6)	41008·0(10)	39431·3(4)	34127·1(3)
M 53405·9.....	46593·3(7)	41457·6(5)	39882·3(4)	
N 51860·9.....	.....	43002·8(6)	41426·1(9)	36122·5(4)
O 53407·7.....	.....	44455·9(4)	42880·2(5)	37575·9(3)
P 47570·8.....	.....	47294·1(7)	45716·0(6)	40410·7(4)
Q 44895·5.....	.....	49969·8(6)	48390·8(8)	43086·0(5)
R 44135·0.....		.....	49152·1(3)	43846·8(5)
S 43141·6.....	.....	51722·3(4)	50145·4(6)	

a consequence of the numerical relations, it is possible to express the wave-numbers of many of the lines as differences between a comparatively small number of terms or energy levels. These terms have also been given in Table II. It is, of course, not possible to assign absolute values to these terms, but the relative values can be given with considerable accuracy. Approximately two-thirds of the lines of Table I. have been included in this term system. The failure to account for the remaining lines has been attributed, in part

at least, to the lack of observational data in the region of the spectrum below λ 1850. The large value of the term separations makes it reasonable to suppose that a number of lines, in the region of the spectrum which has been observed, may be connected by constant difference relations with lines in the Schumann region. For example, wave-numbers calculated from the combinations A–Q, A–R, A–S all fall in the region below λ 1850.

While Zeeman effects have been observed for some of the lines, the resolution obtained is not sufficiently high to enable any of the terms to be identified with certainty. It is possible, however, by a consideration of the general form of some of the patterns, to gain some information as to the relative values of the inner quantum numbers. In their recent work, 'Zeemaneffekt und Multiplettstruktur,' Back and Landé have formulated the following rule, which experiment has shown to be generally valid for a system of odd multiplicities: "For the Zeeman pattern given by the combination of two terms X and Y, the central component has zero intensity when $j_x = j_y$." By the application of this rule to a number of the Zeeman patterns of lines involved in Table II., it may be deduced that the terms C and D have the same inner quantum number. By assigning an arbitrary value to this quantum number, relative quantum numbers may be deduced for a number of the other terms. This method has not so far led to the identification of any complete multiplet groups, but it should prove useful when the required additional wave-length data become available.

Since the terms of the Ag II spectrum have not as yet been definitely identified, it is scarcely possible to make a very useful comparison between this spectrum and that of the neutral atom of palladium. In the present state of our knowledge of the subject, it seems scarcely justifiable to make the assumption that the arrangement of terms in the two spectra will be precisely analogous. If such an assumption were made, however, the following would appear to be the most probable arrangement of the larger terms of Table II. The term D should correspond with the palladium term $^1D_2$. The terms A, B and C would make up a partially-inverted triplet D level in which $A = {}^3D_3$, $B = {}^3D_1$, and $C = {}^3D_2$. The lines formed by combinations with the fundamental ^1S level would probably fall in the Schumann region, and so would not come within the range of the present investigation. Too much importance should not be attached to this suggestion, as the evidence in favour of it is far from complete and the non-appearance of the combinations D–M and D–S is against it.

In considering other points of similarity with the palladium spectrum, it is perhaps worth while to mention that many of the Ag II lines are strongly reversed when observed under spark conditions, and also that they appear with considerable intensity in a flame arc produced *in vacuo*. This indicates that they are "low-level" lines, and as such they correspond with those lines of palladium which have been investigated by the present writer. It may further be noted that the Ag II lines, as compared with those of neutral palladium, show the displacement toward the ultra-violet and the increased value of the term separations which observation of analogous arc and spark spectra of other elements would lead us to expect.

Summary.

New observations of the spectrum of silver have been made with the object of discovering any regularities which may occur among the lines corresponding to a single ionization of the silver atom. The work has included a study of the spectrum under varying intensities of excitation. Zeeman measurements have also been made for a number of the lines.

Thirty lines in the region λ 1850–2930 have definitely been assigned to the spectrum of Ag II. Of these, somewhat over two-thirds have been included in a system of constant wave-number differences involving terms which show some evidence of analogy with the terms found in the arc spectrum of palladium.

The work has served to emphasize the need for an extension of the observations of the silver spectrum to the extreme ultra-violet region of the spectrum. It seems probable that a clear understanding of the Ag II spectrum must await new wave-length data in the Schumann region. Such data should include, not only actual measurements of wave-length, but also a careful comparison of intensities in the arc and spark. It is to be regretted that lack of such observations has made the present investigation incomplete. The results which have been obtained should, however, provide a sound basis for further research, and it is confidently expected that the constant difference relations which have been discovered will provide the key for a complete analysis of the Ag II spectrum.

In concluding the paper the writer wishes to express to Professor Fowler his appreciation of the opportunity of carrying out the experiments, and to thank him also for kind interest and encouragement during the progress of the work.

LXIX. *A Simple Integraph.*

By Alfred A. Robb, *Sc.D., F.R.S.**

If a sharp rounded edge, such as the end of a knife-blade, is pressed against a plane surface of paper it can move freely in a direction tangential to itself, but not freely in a transverse direction.

With suitable guidance such an edge can be made to trace out a curve on the sheet of paper, which can quite clearly be seen as an indentation in the latter. A small sharp-edged wheel turning on an axle may, with advantage, be substituted for the knife-edge.

This principle may be taken advantage of in the construction of an exceedingly simple integraph in the following way.

A small drawing-board is mounted on three supports forming a sort of little table—one of these supports being a rounded knife-edge or a small sharp-edged wheel, as above described, and the others being smooth rounded studs which can move freely in any direction over a flat surface of paper.

For convenience I shall refer to this part of the apparatus as the "*table*," to distinguish it from a larger drawing-board upon which the "table" is placed, and which I shall refer to briefly as the "*drawing-board*." This latter is simply an ordinary drawing-board of suitable size.

To the drawing-board is clamped an arrangement bearing a tracing point of some sort, which remains fixed with respect to the drawing-board, so that the "table," while sliding over the drawing-board, can move freely underneath the tracing point.

Let us suppose now that a curve is drawn on a piece of paper and that the latter is fixed by means of drawing-pins to the "table."

Let us further suppose that a plane sheet of paper is fixed on the surface of the drawing-board and the "table" placed on the top of it.

A suitable weight is put on the "table" so that the wheel may make a visible indentation on the paper below it when the "table" is caused to slide over the latter.

The "table" is now guided so that the tracing point traverses the curve. When this is done the wheel will trace out a second curve, and the relation between the two curves is very simple.

* Communicated by the Author.

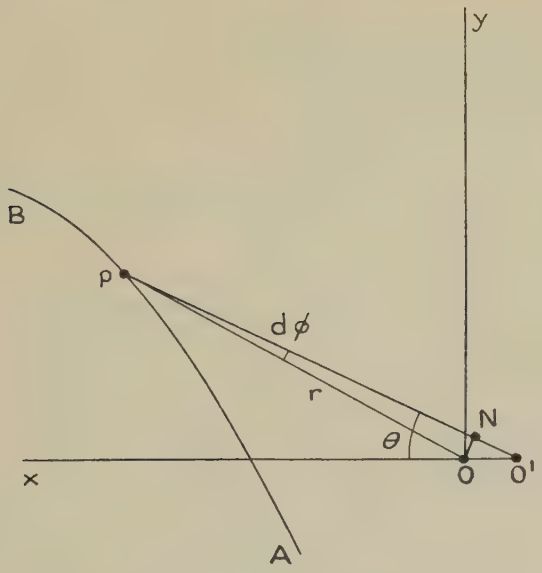
Let AB be the curve on the "table" and P the position of the tracing point upon it at any instant.

Let O be the position of the small wheel and Ox the direction of the tangent to the wheel at its point of contact with the drawing-board.

(The planes of the drawing-board and of the "table" are here treated as coinciding, instead of being parallel as they are in reality.)

The curve traced by the wheel upon the drawing-board will then have Ox as its tangent.

Fig. 1.



Let OO′ be a small element of this curve, and let the small angle OPO′ be denoted by $d\phi$.

Further, let the angle xO′P (which in the limit is equal to xOP) be denoted by θ and the length OP by r.

Let a perpendicular through O upon O′P meet the latter in the point N.

Then O′N$=dr$ and ON$=r\,d\phi$.

Thus

$$r\frac{d\phi}{dr} = \tan\theta.$$

Suppose now that the equation in polar coordinates of the curve on the "table" is

$$\tan\theta = rf(r),$$

where $f(r)$ is an arbitrary function of r; we have then

$$\frac{d\phi}{dr} = f(r),$$

and so $\phi = \int f(r)\,dr + \text{constant}.$

Since P is fixed with respect to the drawing-board, this is the polar equation of the curve traced by the wheel corresponding to the curve

$$\tan\theta = rf(r)$$

upon the "table."

It is to be observed in the figure that the angles θ and ϕ are measured in opposite senses.

Thus in order to integrate $f(r)$ with respect to r we trace the curve

$$\tan\theta = rf(r)$$

upon a piece of paper, which is then fixed to the "table" so that the origin is directly over the point of contact of the wheel and the direction from which the angle θ is measured lies in the plane of the wheel.

The "table" being now moved in such a way that the tracing point traverses this curve, the point of the paper on the drawing-board directly underneath the tracing point is determined, and then the angle between any two radii vectores from this point to the curve traced by the wheel gives the required integral between limits represented by the lengths of these two radii vectores.

The angle is measured by means of a protractor and expressed in radians.

Instead of using r as the independent variable, it is frequently better to employ $\log r$.

If $\log r = z$ we have

$$\frac{r\,d\phi}{dr} = \frac{d\phi}{dz} = \tan\theta.$$

If then we take the curve on the "table" of the form

$$\left.\begin{aligned}\tan\theta &= \mathrm{F}(z)\\ r &= e^z\end{aligned}\right\}$$

we have

$$\frac{d\phi}{dz} = \mathrm{F}(z),$$

and so $\phi = \int \mathrm{F}(z)\,dz + \text{constant}.$

There are two advantages in this.

In the first place we very frequently wish to obtain the integral of a function with the lower limit zero and our method ceases to give accurate results when r is small. If, however, we take z as the independent variable we avoid this difficulty; for since $r=e^z$, we may put $z=0$ and leave r with the finite value unity.

In the second place, provided we have a table of the exponential function, it is generally easier to trace the curve

$$\left.\begin{aligned}\tan\theta &= \mathrm{F}(z)\\ r &= e^z\end{aligned}\right\}$$

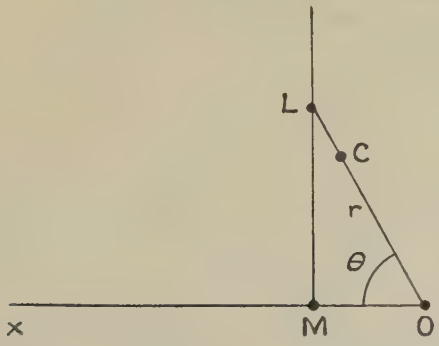
than it is to trace the curve

$$\tan\theta = r\,f(r),$$

since the latter involves multiplying the function to be integrated by the independent variable.

It is in fact practically as easy to plot the first of these curves upon squared paper as it is to plot $\mathrm{F}(z)$ in cartesian coordinates.

Fig. 2.



Thus taking O as origin and any convenient length OM as a unit we can measure lengths in this unit along a line through M perpendicular to Ox which are numerically equal to the values of $\tan\theta$.

Thus to plot the curve

$$\left.\begin{aligned}\tan\theta &= \mathrm{F}(z)\\ r &= e^z\end{aligned}\right\}$$

we select any value of z and take a point L in the perpendicular to Ox through M such that $\mathrm{ML}=\mathrm{F}(z)$ in the selected units.

Then $\mathrm{M\hat{O}L}=\theta$, and we have only to measure off a distance OC along OL such that $\mathrm{OC}=e^z=r$ in order to obtain a point C on the required curve.

If, on the other hand, we wished to plot the curve

$$\tan\theta = r f(r),$$

we should first select a value of r and take $ML = rf(r)$ and then measure off a distance $OC = r$ along the line OL. It is obvious in both cases that the unit in which r is measured along the radius vector need not necessarily be equal to OM, since we are only concerned with the ratio $\frac{ML}{OM}$. It is the unit in which r is measured along the radius vector which gives the scale of the curve.

In using the instrument it is important to know approximately how the "table" is to be rotated so as to keep the tracing point on the curve without pushing the wheel so that it moves sideways.

If a line Oy be taken through O perpendicular to Ox, then it is easily seen that the centre of instantaneous rotation of the surface of the "table" lies at the intersection of Oy with the normal to the curve at P.

When this is kept in mind the "table" can be moved in the proper way without difficulty.

The following figures show the degree of accuracy which may be obtained with such an instrument in integrating the functions $\int \frac{dr}{r}$ and $\int \frac{\sin r}{r} dr$:—

$$\int_1^r \frac{dr}{r}.$$

r.	ϕ in radians.	$\log_e r$ from tables.	Difference.
1	0	0	0
2	0·6836	0·6931	·0095
3	1·0908	1·0986	·0078
4	1·3788	1·3862	·0074
5	1·5999	1·6094	·0095
6	1·7831	1·7918	·0087

$$\int_0^r \frac{\sin r}{r} dr.$$

r.	$\phi + ·199$.	$\int_0^r \frac{\sin r}{r} dr$ from tables.
1·5	1·325	1·325
1·7	1·429	1·449
2·0	1·578	1·605
2·2	1·669	1·687
2·5	1·770	1·778
3·0	1·853	1·849

In the latter case a constant of integration ·199 was taken so as to make the result agree with the tables for $r = 1·5$.

LXX. *Secondary Radiations produced by Gamma Rays and their Effect upon Gamma-Ray Absorption Measurements.*

By L. H. Clark, *Ph.D.**, *M.Sc.*, *A.Inst.P.*, *The Physics Department, Middlesex Hospital* †.

IN 1909, Tuomikoski ‡ investigated the absorption of gamma rays in lead and found that, as the thickness of lead was increased from 0·4 to 18 cm., the absorption coefficient decreased in value from 0·70 to 0·25 cm.$^{-1}$. Recently, many observers have studied the scattered radiation which all matter emits when irradiated by gamma rays, and Owen, Fleming, and Fage § have shown that it may play an important part in absorption measurements. The present investigation was made with a view to gaining more information about these scattered radiations and their effect upon gamma-ray absorption measurements. The experiments are divided into two groups. Those in the first group relate to a study of the radiation entering an electroscope suspended in enclosures of different substances when the direct beam of gamma rays to the former is heavily filtered by lead. In the second group, the effect of radiation scattered from the filters themselves upon absorption measurements is considered.

Fig. 1 is a scale diagram of the experimental arrangement. The electroscope was made of brass 0·065 cm. thick, and was spherical, measuring 2·2 cm. in diameter. It was suspended on a system of piano-wires, the rigidity of which was increased by a light wooden triangular framework D E F. A lead annulus 2·5 cm. in diameter and 0·18 cm. thick served to support the electroscope and a lead cap which covered the latter. The cap was a hollow cylinder of lead 0·18 cm. thick, closed at the top by a lead disk of the same thickness. Another cap of the same dimensions, but with a portion of the cylindrical side removable, was used for some experiments. When in use this cap, which is shown in fig. 1 A, was placed over the electroscope so that the removable portion, termed the window, faced the source of radiation. Measurements were made by timing the rate of fall of the gold-leaf of the electroscope, the case of which was earthed. The gold-leaf

* Part of a Thesis submitted to the University of London for this Degree.

† Communicated by Prof. Sidney Russ, D.Sc.

‡ Y. Tuomikoski, *Phys. Zeitschrift*, 1909, p. 372.

§ A. E. Owen, N. Fleming, and W. E. Fage, Proc. Phys. Soc. xxxvi. 5, p. 355 (1924).

was charged by the movable wire C (fig. 1), which was put momentarily in contact with one pole of a constant voltage supply of 150–170 volts. During measurements, the wire C was earthed.

Fig. 1.

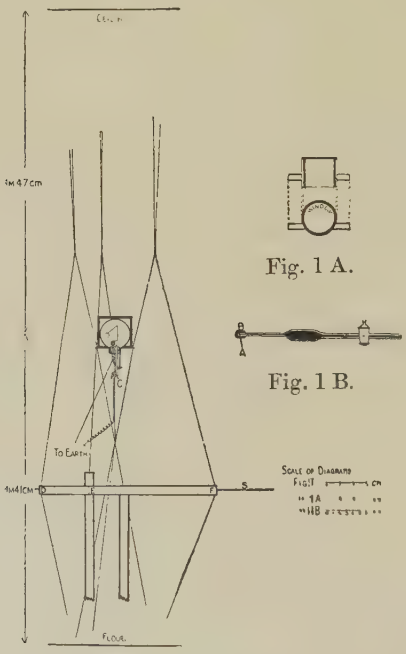


Fig. 1 A.

Fig. 1 B.

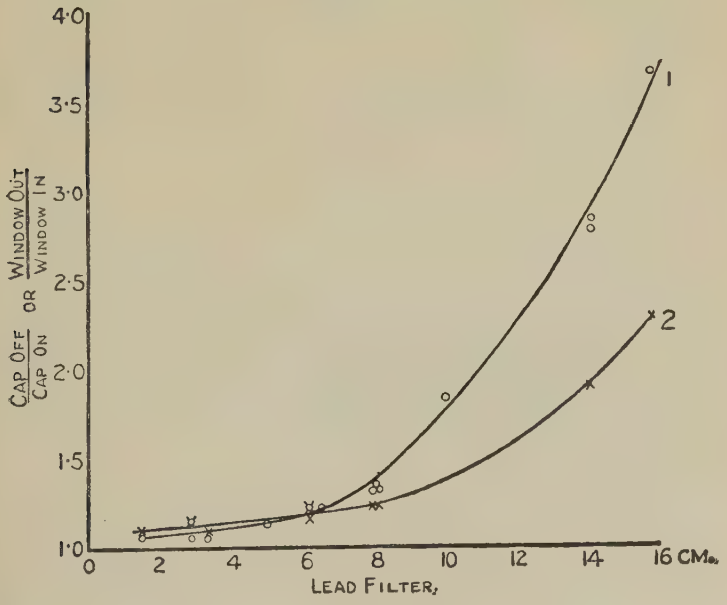
The sources of radiation were fairly large quantities of radium emanation. Fig. 1 B shows the glass vessel used to contain the emanation, so made that it could be filled repeatedly. The emanation was confined to the small bulb A, which was 0·5 cm. in external diameter, the remainder of the vessel being filled with mercury. The bulb A was covered by a small lead cap, the thicknesses of the closed end and sides of which were respectively 0·41 and 0·28 cm. In all the experiments the radiation was therefore first screened by this lead hood.

Group 1.

Preliminary experiments were made with the electroscope suspended mid-way between the floor and ceiling of the room

The radioactive source was set on a level with the centre of the electroscope, and the effect of covering the latter with the lead cap was determined. The ratio of the ionization readings, $\frac{\text{cap off}}{\text{cap on}}$, was 1·067 for distances between the source and the electroscope ranging from 9 cm. to 28 cm. When lead filters were interposed in the primary beam, the ratio $\frac{\text{cap off}}{\text{cap on}}$ was found to increase with the thickness of the former in the manner shown by the curve 1 of fig. 2.

Fig. 2.



The filters were mainly cylinders varying in diameter from 2·37 to 3·55 cm., and the distance of the source from the electroscope ranged from 9 cm. to 28 cm. The curve shows that as the thickness of the filter is increased, the radiation entering the electroscope becomes softer, the change in quality being most marked for filters over 8 cm. thick.

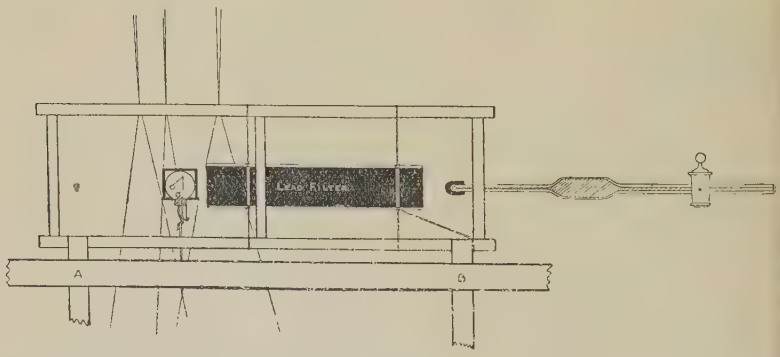
A similar series of measurements was made with the lead cap shown in fig. 1 A. In this case, the major portion of

the cap covered the electroscope throughout the experiment, and the ratio $\frac{\text{window out}}{\text{window in}}$ was obtained for lead filters of different thickness. The variation of this ratio with the thickness of the filter is shown by curve 2 of fig. 2. A comparison of curves 1 and 2 shows that the effect of surrounding the major portion of the electroscope with the second cap is to decrease the quantity of soft radiation entering the electroscope. One may conclude, therefore, that the soft radiation is scattered mainly from the surroundings in which the measurements are made.

Experiments were then made to determine how the quality of the scattered radiation varied with the surroundings in which the measurements were made. The procedure was as follows:—

A series of enclosures of equal internal volume but having walls of different substances were built in turn around the

Fig. 3.



electroscope and source. For this purpose the light wooden framework shown in fig. 3 was erected around the apparatus. Surrounds of aluminium, zinc, lead, 3-ply wood, paraffin wax, and brick were made by covering the sides of the framework with the different substances. The direct beam of gamma rays to the electroscope was practically eliminated by a cylindrical lead filter 15·86 cm. long and 3 cm. in diameter. The distance between the source and the centre of the electroscope was maintained approximately 20 cm. The quality of the radiation entering the electroscope was measured by the ratio $\frac{\text{cap off}}{\text{cap on}}$.

Table I. gives the results obtained with the different enclosures and includes the values found when the wooden framework was left bare so that the room formed the surround.

TABLE I.

Surround.	Wall thickness.	% Absorption. Primary γ rays.	Cap off. Corrected reading.	$\frac{\text{Cap off}}{\text{Cap on}}$
Room	—	—	1·46	4·34
Aluminium	0·013 cm.	—	1·63	5·49
Zinc	0·023 „	0·6	2·56	4·28
3-ply Wood	0·50 „	—	2·65	3·54
Lead	0·18 „	8·0	5·08	3·18
Paraffin Wax ...	2·7 „	6·2	7·39	3·37
Brick.....	6·4 „	29·7	28·0	4·93

The ratio $\frac{\text{cap off}}{\text{cap on}}$ for the different surrounds shows no marked variations, but the "cap off" readings differ considerably among themselves. The thickness of the enclosure wall varied from one substance to another, and a comparison of the figures in columns 3 and 4 shows that the largest "cap off" readings were obtained with those surrounds, the walls of which absorbed the largest quantities of primary gamma radiation. This is further shown by the figures in Table II.,

TABLE II.

Lead Surround.	Ionization Readings.		$\frac{\text{Cap off}}{\text{Cap on}}$
	Cap off.	Cap on.	
0·17 cm.	4·23	1·38	3·06
0·33 „	4·56	1·7	2·68
0·66 „	4·99	2·0	2·49

which were obtained for lead enclosures of equal internal volume but differing in wall thickness. This Table shows in addition that the scattered radiation from the lead enclosure

becomes more penetrating as the thickness of the walls is increased.

To compare the quality of the radiation scattered from different substances, the above results indicate that thicknesses of the various substances should be chosen which are equivalent in absorbing power for the primary gamma rays. The ratio $\frac{\text{cap off}}{\text{cap on}}$ for enclosures having equivalent wall-thicknesses of lead and of brick of 0·66 and 6·4 cm. were respectively 2·49 and 4·93 cm.$^{-1}$. The coefficients of absorption in lead of these scattered rays are respectively 5·06 and 8·85 cm.$^{-1}$. Assuming Owen's law, viz. $\mu \propto \lambda^3$, where μ and λ have their usual significance, it can be shown that $\frac{\lambda \text{ brick}}{\lambda \text{ lead}} = 1{\cdot}2$. The internal volume of the brick surround was approximately 38 per cent. larger than that of the lead enclosure. Subsequent experiments show that the difference in wave-lengths of the scattered radiations from brick and lead surrounds of equal internal volume would be less than 20 per cent. These scattered radiations have wave-lengths approximating to those of X-rays excited by a peak potential of 196,000 volts, for which Rutherford * found a mass absorption coefficient of 0·75 corresponding to a wave-length of 0·063 Å unit. Hull found that $\frac{\mu}{\rho}$ for the K characteristic radiation of lead in lead was 1·51, denoting a less penetrating radiation than the observed scattered radiation.

The above results are in general agreement with those of Compton †, who showed by a different method that matter irradiated by gamma rays emits a softer radiation, the wave-length of the softest components of which lay between 0·06 and 0·12 Å unit.

A number of enclosures of different internal volume having walls of the same material and thickness were then built around the electroscope, and the quality of the radiation scattered from each was measured in the manner of the last experiment. Table III. gives the results obtained with brick enclosures.

This material was chosen so that the surrounds would consist of approximately the same substance as the walls of the room. The figures within brackets in the first column

* E. Rutherford, Phil. Mag. xxxiv. p. 153 (1917).
† A. H. Compton, Phil. Mag. xli. p. 749 (1921).

TABLE III.

Brick Surround.		Distance Source – Electroscope.	Ionization Readings.		Cap off / Cap on.
Number of Bricks.	Internal Volume.		Cap off.	Cap on.	
12	6·28 litres	19·9	20·3	3·96	5·12
42	42·6 „	„	23·7	3·12	7·6
Room	—	„	1·4		
12	6·28 „	19·8	15·9	3·23	4·93
Room	—	—	0·80		
19	7·74 „	20·7	8·82	1·44	6·12
31	24·8 „	—	9·46	1·37	6·9
26	26·3 „	21·2	19·0	2·82	6·74
49	75·0 „	20·5	20·9	2·1	9·83

were obtained on the same day with the same source, so that the ionization readings thus bracketed may be compared. For these, it is seen that the "cap off" readings increase but the "cap on" readings diminish with increase in the size of the enclosure. The ratio $\frac{\text{cap off}}{\text{cap on}}$ thus increases considerably as the enclosure is enlarged.

TABLE IV.

Surround.	Distance Source – Electroscope.	Ionization Readings.		Cap off / Cap on.
		Cap off.	Cap on.	
Large Lead ...	20·0	5·48	0·85	6·45
Small Lead ...	19·8	5·08	1·6	3·18
Room.....	„	1·33		

The results given in Table IV. lead to similar conclusions. They were obtained when the apparatus was surrounded by lead enclosures of wall thickness 0·18 cm. The internal volumes of the large and small enclosures were respectively $55{\cdot}5 \times 44 \times 44$ cm. and $11 \times 12 \times 34{\cdot}5$ cm.

A consideration of the above results shows that the change in volume of a lead surround required to double the ratio $\frac{\text{cap off}}{\text{cap on}}$ is approximately twice that necessary to produce the same change with brick enclosures. These facts indicate that the radiation entering the electroscope is complex. If a single scattered radiation from the walls of the enclosures was present, one would expect that both the "cap off" and "cap on" readings would diminish as the surround was enlarged, and not as has been found. The increase in the ratio $\frac{\text{cap off}}{\text{cap on}}$ with increase in the size of the surround, and therefore with the quantity of air enclosed by the latter, suggests the presence of a scattered radiation from the air which is softer than that from the walls of the enclosure. Experiment has shown that the radiation scattered from a brick surround is softer than that from one of approximately the same size of lead 0·18 cm. thick, the ratio $\frac{\text{cap off}}{\text{cap on}}$ for the former being 1·55 times that for the latter. Moreover, it has been seen that the quantity of scattered radiation is intimately related to the amount of primary radiation absorbed in the scattering substance. Assuming the presence of a scattered radiation from the air, one would expect therefore that, in order to decrease the penetrating power of the radiation entering the electroscope by a given amount, the necessary change in volume for the lead surround must be greater than that for the brick enclosure. The experimental data support this conclusion.

Group 2.

In this section, some experiments are recorded showing the effect of radiation scattered from the filter upon gamma-ray absorption measurements.

Conical lead filters were made of varying length having apex angles equal to the cone of rays from the source which just enveloped the bulb electroscope in the particular experiments. The amount of radiation scattered from the filter would thus be reduced to a minimum, and one would expect that lower transmission values than those of Tuomikoski* would be obtained. From the fignres given in Table V., however, it is seen that the values of Tuomikoski are in all cases lower. This result indicates, therefore, that the

* *Loc. cit.*

amount of scattered radiation from the surroundings which entered the electroscope was greater than in Tuomikoski's experiment.

TABLE V.

Cone Thickness. cm.	Transmission. per cent.	Tuomikoski's Values.
1·33	43·4	42·9
1·45	3·34	3·17
7·08	2·52	2·51
7·95	2·02	1·50

A comparison of the ionization readings obtained with two cylindrical filters 14·0 and 14·1 cm. long, the respective cross-sections of which were in the ratio of 1·5 to 1, showed that the readings for the filter of smaller diameter was the larger by 24 per cent. For two lead filters 6·7 cm. thick, one having a cross-section 25 times the other, the ionization readings differed by 22 per cent., the larger reading in this case being obtained with the filter of larger cross-section.

It has been shown in the first section of the paper that the effect of radiations scattered from the surroundings may be considerable. An increase in the cross-sectional area of a filter will increase the quantity of radiation scattered from the filter but will diminish the amount of scattered radiation from the surroundings which enters the electroscope.

The variation in the apparent transmission of gamma rays through comparatively thin filters with their disposition between the source and the electroscope was then measured. Ionization readings were taken with (*a*) the filter near the electroscope—the A position, (*b*) the filter near the source—the B position, and (*c*) no filter. This experiment was made with cylindrical filters, 3 cm. in diameter, of silver, lead, and platinum. The thickness and mass of each was as follows :—

	Thickness.	Mass.
Silver . . .	1·55 cm.	113·8 gms.
Lead . . .	1·45 ,,	111·5 ,,
Platinum . .	0·75 ,,	112·2 ,,

The results obtained with the respective filters are given in Table VI.

Table VI.

Filter.		Distance from Electroscope to Filter. cm.		Percentage Transmission.	
Element.	Thickness, cm.	A Position.	B Position.	A Position.	B Position.
Silver.....	1·55	2·5	26·2	48·3	55·4
Lead	1·43	,,	,,	42·1	45·6
Platinum ...	0·75	,,	,,	41·9	45·2

The apparent transmission of each filter is greater when it is placed near the source than when near the electroscope, a result previously obtained by Florance *, who used lead filters. For lead and platinum cylinders, the difference in the apparent transmission was approximately 8 per cent., but that for the silver one amounted to nearly 15 per cent. To determine the cause of this variation, the further experiment was made of comparing the transmission of a lead filter placed in the A position with that of another of the same thickness but of smaller cross-section placed in the B position.

Fig. 4.

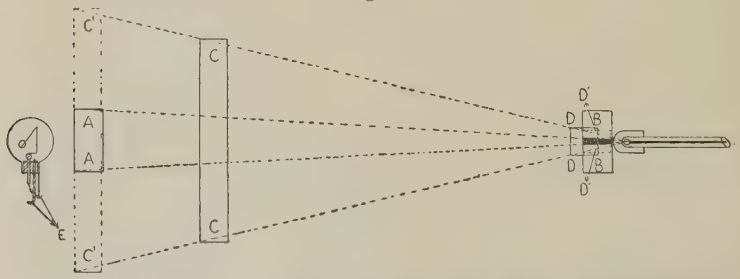
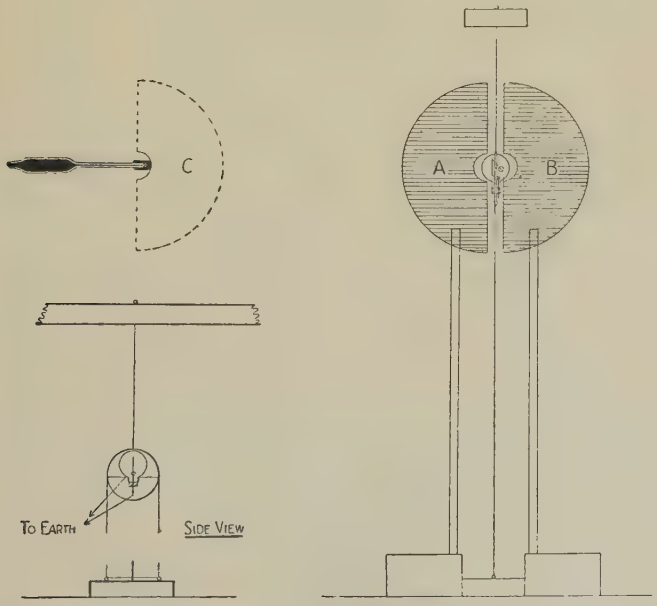


Fig. 4 is a scale diagram of the experimental arrangement, where C and D represent the filters so placed that C coincided with the shadow of D thrown by the source, the area of C being 57·3 times that of D. Each filter thus received practically the same quantity of gamma rays. The apparent transmission of the larger filter was 8·8 per cent. bigger than that of the smaller one, which may be attributed to the

* D. C. H. Florance, Phil. Mag. ii. p. 921 (1910).

excess of scattered radiation from C, which subtended a bigger solid angle at the electroscope than D. It is safe to assume that the difference would be greater for two similar filters in the dotted positions C′ and D′, the areas of which would be in the ratio of 135 to 1. These dotted positions coincide with the two positions A and B of the silver, lead, and platinum cylinders used in the former experiment. The difference between the apparent transmission of the cylindrical lead filter A and the small shaded portion of it in the B position, would be less than that between the filters C′ and D′. The effect of the whole lead cylinder B, the area of which is 73 times that of the small shaded portion, will make the apparent transmission of the filter in this position greater than when the same filter is placed at A. One may conclude therefore that a given filter becomes a more powerful source of scattered radiation the nearer it is placed to the source of primary gamma rays, and consequently appears more transparent in this position.

Fig. 5.



A study was then made of the radiation scattered from filters of paraffin wax. For this purpose a suspended silver sphere electroscope was used to measure the radiation transmitted by wax filters which were hemispherical. The experimental arrangement is shown in fig. 5.

The electroscope was 2·57 cm. in diameter and its wall-thickness 0·0158 cm. The diameters of the filters were respectively 16·3, 10·0, 7·8, and 5·2 cm., each filter being made in duplicate. The radon source used in former experiments was set up at a distance of 33 cm. from the centre of the electroscope. Readings were taken under the following conditions :—

(1) Without wax.
(2) With a paraffin hemisphere in the position A.
(3) With a wax sphere A, B surrounding the electroscope.
(4) With a wax hemisphere in the dotted position C.

Table VII. shows the results obtained.

TABLE VII.

Sphere and Hemisphere.	% Transmission.		% Transmission.	
Diameter, cm.	Hemisphere.	Sphere.	Without Lead.	With Lead.
0	—	—	—	91·0
5·2	99·1	100·	103·8	84·2
7·8	95·5	101·	101	83·0
10·0	94·1	100·5	100·7	81·6
16·3	94·4	107·8	103·8	79·0

The apparent transmission of the filters in the A and C positions are given respectively in columns 2 and 4. The apparent transmission of a wax cylinder 3 cm. in diameter and 7·0 cm. long, which was equivalent in thickness to the largest hemisphere, was 69·3 per cent. when placed in the C position. The corresponding figure for the largest hemisphere was 103·8 per cent. It is clear, therefore, that much of the ionization recorded with the wax hemispheres is due to radiations scattered from them. Moreover, the figures in column 4 indicate that the amount of this scattered radiation increases with the size of the hemisphere. This fact is further illustrated by the figures in column 3, which were obtained when the electroscope was enclosed in turn by wax spheres of increasing diameter.

The quality of the radiation transmitted by the hemispheres placed in the C position was then compared with that of the primary rays. A lead sheet measuring $12 \times 14{\cdot}4$

× 0·16 cm. was mounted vertically at a distance of 4·5 cm. from the electroscope, and its transparency was found for (*a*) the primary rays, (*b*) the rays transmitted by the wax hemispheres. The results thus obtained are shown in the last column of Table VII. The transmission of the lead sheet is seen to decrease with increase in the diameter of the hemisphere, showing that as the size of the latter is increased, the radiation entering the electroscope becomes softer. With the wax cylinder filtering the primary beam, the lead sheet transmitted 90·3 per cent. of the radiation incident upon it. One may conclude that the softening of the radiation with increase in the diameter of the wax hemisphere is due to the increase in the amount of scattered rays, which are softer than the primary gamma radiation. The quality of this scattered radiation may be determined as follows :—The difference in the ionization readings obtained with the largest hemisphere and cylinder was 2·65 div./min., which fell to 1·38 div./min. when the lead sheet was interposed. The absorption coefficient in lead calculated from these figures is 4·06 cm.$^{-1}$, which is of the same order as those obtained for the radiations scattered from the enclosures described in Part I. of this paper.

Conclusion.

Absorption measurements made with an electroscope suspended in air show the presence of a general scattered radiation from the surroundings which may contribute more to the ionization than the radiation transmitted by very thick lead filters. This scattered radiation is considerably softer than the primary rays which provoke it. A study of the radiations scattered from different substances irradiated by the same beam of gamma rays reveals the following facts :—

(1) The penetrating power of the radiation scattered from layers of lead up to 0·66 cm. thick increases with the thickness of the layer.

(2) The radiations scattered from layers of different substances which are equally absorbent for the primary gamma rays have a wave-length of about 0·063 Å unit, corresponding to X-rays excited by a peak potential of 196,000 volts.

Experimental evidence has been obtained of a comparatively soft radiation scattered from air irradiated by gamma rays.

To avoid errors in absorption measurements due to scattered radiations, it is suggested that use should be made of the fact that they are comparatively soft. A lead filter 1·0 cm. thick absorbs nearly 99 per cent. of the hardest scattered radiation dealt with in the above experiments, and the use of an electroscope having walls of this material and thickness would practically eliminate errors due to this cause.

I thank the Medical Research Council for the use of large quantities of radium emanation and for facilities to carry out some of the experiments. To Mr. Spicer, of the Radon Department of the Middlesex Hospital, I am indebted for the preparation of the radon sources.

The experiments were made at the instigation of Professor Russ, and I thank him warmly for the help and valuable suggestions he has given me.

LXXI. *The High Vacuum Arc in Hydrogen.* *By* F. H. Newman, *D.Sc., A.R.C.S., Professor of Physics, University College of the South-West of England, Exeter* *.

[Plate XV.]

IN a previous paper † it has been shown that with metallic sodium as the cathode an arc in hydrogen gas at low pressure can be started and maintained between cold electrodes with comparatively small applied potential differences. At very low gas pressures, *i. e.* before the heat from the arc vaporized the sodium to any appreciable extent, the glass walls of the containing tube showing the green fluorescence under the electric discharge action, the radiation from the arc consisted mainly of the secondary and Balmer series hydrogen lines, the brightest lines being those which constitute the first Fulcher bands in the red. It is well known that high relative intensity of the secondary lines appears under less energetic conditions of excitation and, in particular, the Fulcher bands are developed at low potentials. This type of arc differs from the normal arc in that its cathode is not incandescent or even hot, and therefore

* Communicated by the Author.

† Phil. Mag. i. p. 940 (1926).

thermionic electrons are not supplied continuously in great abundance at the negative pole. Under normal conditions the arc is most difficult to start, and is most intermittent, when formed between metal electrodes which conduct heat well. It is also difficult to maintain in hydrogen, this gas having a higher thermal conductivity than air. In the case of the normal arc the cathode is presumably maintained in the high temperature condition by the collisions on it of positive ions which have been subjected to the cathode potential fall. With cold electrodes other processes must supply the electrons necessary for the arc maintenance. The atoms of the gas, which are in the metastable condition as the result of collisions with electrons, will on returning to the normal state emit radiant energy which may strike the cathode, be absorbed by it, and cause electrons to flow from the cathode. Electrons are also emitted from a cold plate bombarded with positive ions and, in addition, positive ions flowing rapidly across a gas ionize some of the atoms in it. These latter processes, however, are inefficient at the low speeds with which the ions move under low potential differences. Recently Thomson * has pointed out that there are, probably, several types of radiation near the cathode of an electric discharge-tube, and similar radiations may arise within the cathode fall of potential in the arc.

In the present experiments the vacuum arc tube was similar to that used previously, but an iron electrode was substituted for the sodium cathode, so that the arc was struck and maintained between iron electrodes which were distant apart 2 cm. A momentary electric discharge could be passed between one of these arc electrodes and a third one inserted in the tube. This initial discharge was necessary to strike the arc. The radiation was viewed and photographed by means of a quartz spectrograph through a quartz window.

At first the tube was exhausted continuously to a pressure of 10^{-4} mm. of mercury, the residual gases consisting chiefly of oxygen, nitrogen, hydrogen, water vapour, and the carbon oxides. Applying 80 volts across the arc electrodes, and passing an electric discharge through the tube by means of a coil, a brilliant arc through the residual gases was obtained. The arc persisted after the electric discharge from the coil was stopped. The radiation was nearly white in colour, and when examined was found to consist chiefly of the Balmer and secondary hydrogen lines together with a continuous

* Phil. Mag. xlviii. p. 1 (1924).

spectrum throughout the range λ 6000–λ 2000. The nitrogen bands were faint, but the ammonia bands λ 3360, λ 3371 were prominent. The current could be varied between 6 and 10 amperes by means of a rheostat, but at the higher currents the tube soon fractured owing to the intense heat generated. The formation of such an arc is interesting and was unexpected because the ordinary electric discharge passed through the tube with difficulty and there had been no sign of luminosity, the tube walls fluorescing under the cathode ray action. This arc persisted for about a minute. Although the cathode became very hot there was no trace of iron lines in the radiation, so that there does not appear to be any appreciable vaporization of the iron electrodes. The intense heat generated by the arc liberated gases occluded within the glass walls, and if these gases were not removed rapidly the gas pressure rose to such a high value that the arc was extinguished.

Occasionally a brilliant white glow was formed between the arc electrodes. It was momentary, and was certainly not the typical arc. The electrodes remained cold while this glow existed and no occluded gas appeared. It was impossible to produce the glow in the absence of the potential difference across the arc electrodes, in fact the ordinary electric discharge from the coil gave no luminosity within the tube owing to the low gas pressure. The glow spectrum resembled that of the arc, although with the latter the secondary hydrogen lines were relatively brighter than the Balmer lines. This is shown in spectrograms Nos. I and II (Pl. XV).

In a second series of experiments a stream of hydrogen generated by the electrolysis of a phosphorus pentoxide solution was passed through the tube for several hours to eliminate, as far as possible, all gases except hydrogen and water vapour. With the gas at a pressure of 10^{-2} mm. of mercury an arc similar to that described above could be started and maintained at low potential differences between the arc electrodes. The cathode became very hot, almost incandescent, but there was no trace of iron lines in the spectra of the emitted radiation. In some experiments the current was 10 amperes, but the actual fall of potential across the electrodes was never less than 40 volts. Such an arc in hydrogen at the voltages in the absence of an incandescent cathode is unusual.

The arc would strike with the gas pressure at 10^{-1} mm. of mercury if the iron electrodes were clean and bright, but

after the first flash had been extinguished it was very difficult to restart the arc unless the electrodes were removed and the surfaces cleaned. A probable explanation of this is the formation of an iron-oxide film on the electrodes. Oxygen liberated from the dissociated water vapour, which was always present, acts on the iron forming the oxide. This film would increase the cathode fall.

Both the arc and glow in hydrogen could be obtained at pressures below 10^{-3} mm. of mercury, and although the spectra of the radiation in the two phenomena were similar, the arc spectrograms indicated more continuous spectrum than those of the glow, and the water bands in the ultraviolet were more prominent in the glow radiation. Greater relative intensity of the Balmer lines and disappearance of the continuous spectrum were associated with prominent water bands, as shown in spectrograms Nos. III, IV, V (Pl. XV.).

The initial ionization of the gas molecules when this high density current arc is started must be produced indirectly by the cathode rays. As pointed out by Thomson *, the source of ionization in the Crookes' dark space is not due wholly or mainly to the detachment of electrons from gas molecules by collision. In the present experiments the dark space extended throughout the tube, and it is probable that the ionization of the gas molecules is brought about by high-frequency radiation from those gas molecules which have been subjected to collisions with cathode rays. These collisions do not result in the splitting off of electrons, but the electrons in the inner layers of the molecules are displaced without ejection from the molecule. The return of these electrons to their old positions would be accompanied by the emission of high-frequency radiation which could ionize molecules on which it fell. The recombination of the ions produces the glow, and the absence of any strong field, *i. e.* when the coil discharge ceases, allows such recombinations, the gas glows and finally arcs.

* *Loc. cit.*

LXXII. *On the Action in Parabolic Paths under Gravity.* *By* W. B. Morton, *M.A., Queen's University, Belfast* *.

A CERTAIN amount of what may be called arithmetical vagueness is attached to the Principle of Least Action owing to the lack of numerical values for the magnitude in question. Analytical expressions for the Action can be found in simple cases of free paths, but in order to illustrate the stationary property possessed by such values one would like to follow the variation of the Action when the free path is gradually deformed, in some definite way, into adjacent guided paths. I have done this for the parabolic trajectory under gravity, between two points at the same level, the varied paths being also parabolic. The action between two points on a guided parabolic path with axis vertical, can be expressed by means of elliptic functions. The results have perhaps some interest as illustrating the Principle of Least Action and also a point in the calculus of variations.

A particle is projected along a smooth tube, or a bead along a smooth wire, in the form of a parabola with axis vertical and vertex upwards. Let c be the height of the level of no velocity above the vertex of the parabola $y^2=4ax$, then the action between the ends of a horizontal chord is

$$A=2\sqrt{(2g)}\int_0^x (x+c)^{\frac{1}{2}}.(x+a)^{\frac{1}{2}}.x^{-\frac{1}{2}}dx.$$

This is an algebraic function of x in two cases :

(1) when $c=a$ and the path is a free trajectory,

$$A=\frac{4}{3}\sqrt{(2g)}x^{\frac{1}{2}}(x+3a)\ ;$$

(2) when $c=0$ and the path touches the level of no velocity,

$$A=\frac{4}{3}\sqrt{(2g)}(x^{3/2}-a^{3/2}).$$

In other cases A is expressed by elliptic functions.

Suppose $c>a$, then the substitution

$$x=a\,sc^2u \text{ with } k^2=(c-a)/c$$

transforms the integral into

$$2ac^{\frac{1}{2}}\int dn^2u\,.\,nc^4u\,.\,du=\frac{2}{3}ac^{\frac{1}{2}}\{snu\,.\,dnu\,.\,nc^3u + 2u+(2-k^2)(snu\,.\,dnu\,.\,ncu-Eu)/k'^2\}.$$

* Communicated by the Author.

The case when $c<a$ is met by interchange of letters c, a. If it is desired to include paths lying below the horizontal with the vertex of the parabola downwards, the action is

$$2\sqrt{(2g)}\int_0^x (c-x)^{\frac{1}{2}}.(x+a)^{\frac{1}{2}}.x^{-\frac{1}{2}}dx.$$

The substitution $x=c\,cn^2u$ with $k^2=c/(a+c)$ gives for the above integral the expression

$$2c(a+c)^{\frac{1}{2}}\int sn^2u\,.\,dn^2u\,.\,du$$
$$=2c(a+c)^{\frac{1}{2}}\{k'^2u-k^2snu\,.\,cnu\,.\,dnu+(2k^2-1)\mathrm{E}u\}/3k^2.$$

For comparison with these parabolic paths we may include the action along paths made up of two vertical lines connected by a horizontal line at their upper ends. If the initial upward velocity is that due to height h, the height of the horizontal part of the path z, and the horizontal range r, then

$$\mathrm{A}=\sqrt{(2g)}\left\{2\int_0^z (h-x)^{\frac{1}{2}}dx+(h-z)^{\frac{1}{2}}r\right\}$$
$$=\frac{4}{3}\sqrt{(2g)}\left\{h^{3/2}-(h-z)^{3/2}+\frac{3}{4}(h-z)^{\frac{1}{2}}r\right\}$$

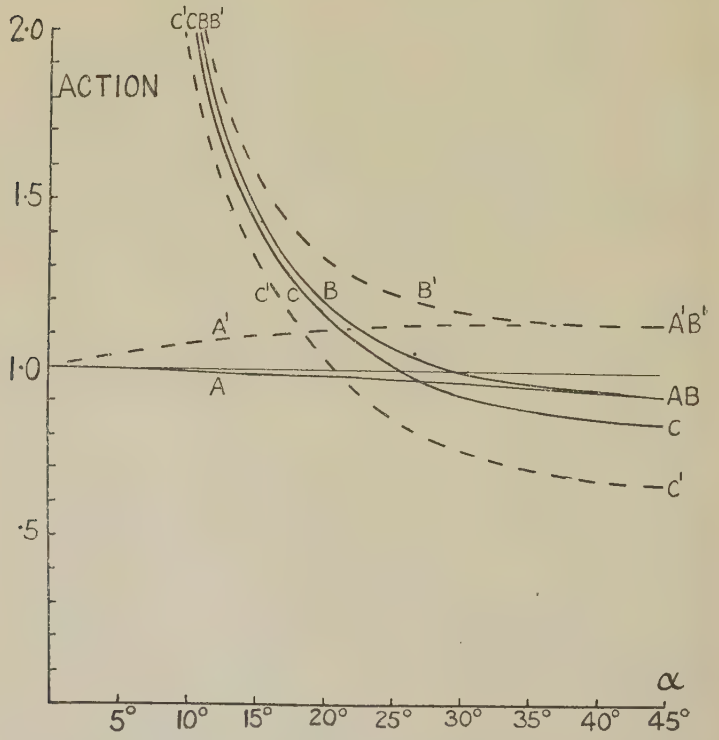
with the limiting value $\frac{4}{3}\sqrt{(2gh^3)}$.

It is convenient to take as unit of action the value for the direct passage at constant velocity along the horizontal range. This is $4\sqrt{(2gax\,.\,c+x)}$ in the parabolic, and $\sqrt{(2gh)}r$ in the rectangular paths.

With a given velocity of projection $\sqrt{(2gh)}$ each horizontal range r, inside the maximum $2h$, can be reached with a pair of complementary angles of elevation $\left(\alpha, \frac{\pi}{2}-\alpha\right)$. We shall take $\alpha<\frac{1}{4}\pi$. Then $r=2h\sin 2\alpha$. The two trajectories may be distinguished as the lower and upper free paths. Their vertices are at heights $h\sin^2\alpha$, $h\cos^2\alpha$, and their parameters are $h\cos^2\alpha$, $h\sin^2\alpha$, respectively. The formulæ given above enable us to calculate, without elliptic functions, the values of the action for three special parabolic paths, viz., the two free paths and the "limiting parabola" which touches the level of no velocity, and for three rectangular paths whose horizontal portions are at the level of the vertices of the three parabolas.

The results are plotted (fig. 1) against the angles of elevation, the unit in each case being the horizontal action for the corresponding range. The full lines are for the parabolas, and the broken lines for the rectilinear paths. Curve A shows the action for the lower trajectory, B for the upper, and C for the parabola which rises to the level of no velocity. A′B′C′ are for the corresponding rectangular paths.

Fig. 1.

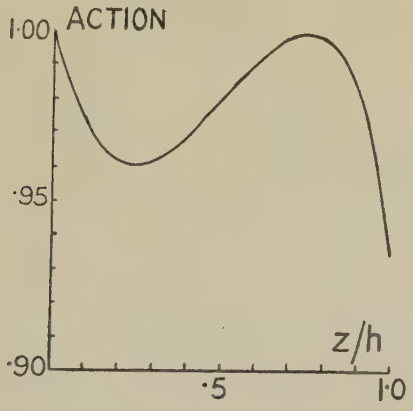


Let now a particular range be taken, and let a series of parabolas be drawn with vertices at heights ranging from zero to h. The parabolas which lie below the lower and above the upper free paths have their directrices above the level of no velocity ($a > c$), those between the two free paths have their directrices below this level ($a < c$). It is known that the action along the lower trajectory is a true minimum because the end of the path is reached before the point where the path touches the parabolic envelope of all the

parabolas which can be described from the initial point with the same velocity of projection. This point, the "kinetic focus," is at the other end of the chord drawn from the point of projection to the focus. For the upper trajectory, on the other hand, the kinetic focus is passed during the flight and the action is a minimum for some ways of deforming the path but a maximum for others.

The calculation of the formula involving elliptic functions has been carried for the case $\alpha = 30°$. The modular angle was given a series of values, multiples of 5°, and the action tabulated against z/h, the height of the vertex expressed as a fraction of the velocity-head. The result is shown on fig. 2.

Fig. 2.

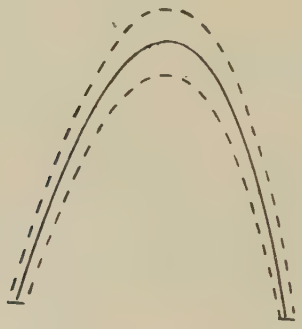


It will be seen from fig. 1 that for this special value of α the action along the upper trajectory is equal to that along the horizontal range which has been taken as unit. It appears that for this kind of variation there is a maximum value of the action along this free path. The minimum action along the lower trajectory is only about 4 per cent. less than this, so the ordinate has been plotted on a large scale beginning from the value ·90.

One would like to complete the illustration of the difference between the two kinds of free path by devising some other way of gradually varying the curves which would give minimum action for the upper trajectory also. A way which suggests itself, and which retains the parabolic form, is to lower or raise the free path without changing it otherwise, introducing horizontal straight pieces at the ends to

complete the journey (fig. 3). Beginning with either free path we may depress the vertex down to the horizontal range or raise it to the level of no velocity. There is a fundamental difference, however, between this way of changing the path and the previous one. No matter how close the changed path is to the free trajectory, it is not a "slightly varied path" in the meaning of the calculus of variations. In the procedure of that calculus it is assumed that not only the ordinate but also the slope of the curve is varied continuously, that both δy and $\delta(dy/dx)$ are small quantities of

Fig. 3.



the first order. The latter condition is not fulfilled at the ends of the curve in the present case, and, in consequence, the change in the action is of the first instead of the second order. Notwithstanding this, the action along either free path possesses the minimum property, in the sense that it increases when we pass to the changed path on either side. If the vertex of the parabola is raised by δz, the action is increased by

$$\sqrt{(2ga/x)}\{(x+a)^{\frac{1}{2}}+a^{\frac{1}{2}}\}\delta z\ ;$$

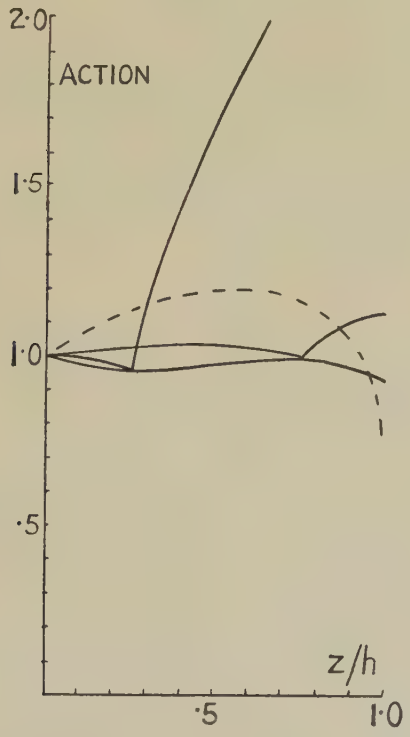
if it is lowered, there is again an increase of

$$\sqrt{(2ga/x)}\{(x+a)^{\frac{1}{2}}-a^{\frac{1}{2}}\}\delta z.$$

The discontinuity is, of course, due to the fact that the contributions of the little straight pieces are always positive. To get analytical continuity in the values they should change sign in passing from one side to the other of the end-points.

These results are shown on fig. 4. The action graph of fig. 2 is plotted on one-tenth the scale, and from its minimum and maximum there spring pairs of curves showing the action for the discontinuous paths. The broken line has

Fig. 4.



been added to show the action along the upward and downward verticals connected by a horizontal line at level z. This has a maximum value when $z=h-\frac{1}{4}r$ when, in terms of the unit adopted,

$$A=\frac{4}{3}(h/r)+\frac{1}{3}\sqrt{(r/h)}.$$

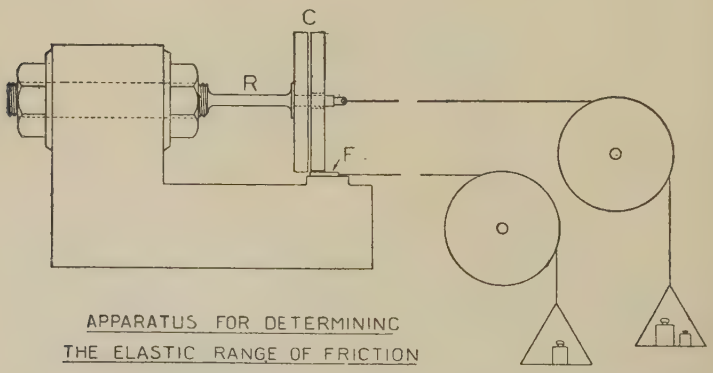
In the present case $r=h\sqrt{3}$, which gives for the maximum

$$z/h=\cdot 567,$$
$$A=1\cdot 209.$$

LXXIII. *The Elastic Range of Friction.* *By* James S. Rankin, *B.Sc., A.R.T.C., Lecturer, The Royal Technical College, Glasgow* *.

IN a paper on "Molecular Contact" by James S. Stevens, published in the 'Physical Review' for 1899, it is shown (by an interferometer method) that when one metal plate is pulled over the surface of another, as in an ordinary experiment for finding coefficients of friction, then, before slipping occurs, *elastic* strains are produced of magnitudes one or two tenths of a wave-length of yellow light ($\lambda = \cdot 00006$ cm.). It was thought that a further investigation of this elastic range of friction, using an oscillating-valve ultramicrometer method capable of detecting say 10^{-7} cm.†, might lead to

Fig. 1.



APPARATUS FOR DETERMINING THE ELASTIC RANGE OF FRICTION

a better understanding of the nature and distribution of ordinary friction and of the action of the surface films concerned.

The apparatus first employed is illustrated by figs. 1 and 2. A parallel plate condenser C, formed of two metal plates (7 cm. $\times 2\frac{1}{2}$ cm.) with surfaces say 1/20 mm. apart, was placed in an oscillating electric circuit (frequency of the order one million per second) connected to the grid of a thermionic valve. A second valve was provided with a similar grid circuit having a variable condenser, so that its frequency could be adjusted to say 1 million 200 per second. The high-frequency currents in the plate or sheath

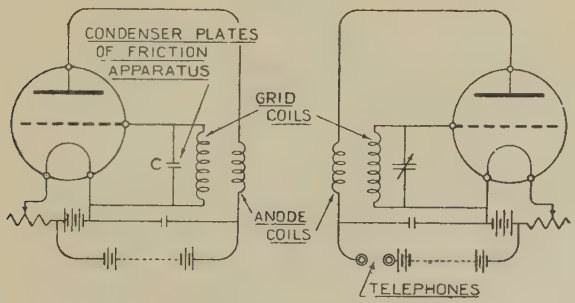
* Communicated by the Author.

† Whiddington, Phil. Mag. Nov. 1920. He detected 5×10^{-8} cm.

circuits of these two oscillating valves, acting on ear telephones, gave rise, due to interference, to an audible note in the telephones of beat frequency about 200 per second. An alteration of the capacity of the condenser C, produced by a change in the distance between the plates of say 10^{-7} cm. caused the frequency of the note heard in the telephones to alter in a manner which could readily be recognized by the ear.

One of the plates of the condenser C—the right-hand plate which had a hole drilled in its centre—was fixed to the top of the thin plate F ; the friction at the under surface of F being the subject of investigation. The other plate of the condenser C was attached to a carefully measured standard steel rod R fixed to a rigid support ; the dimensions of the rod were such that a pull of 100 grams extended

Fig. 2.



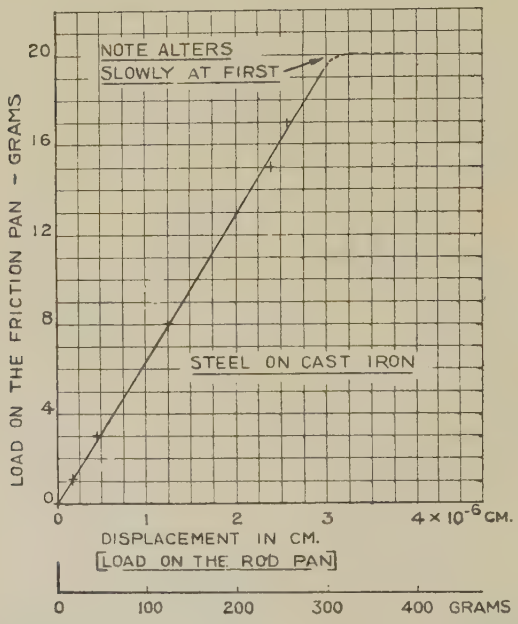
OSCILLATING VALVE CIRCUITS FOR FRICTION APPARATUS

it by 10^{-6} cm. The apparatus was enclosed in a lead-lined box to avoid electric induction effects and placed on a firm foundation to avoid vibration.

A tangential pull (of say 10 grams weight) applied to the plate F slightly increased the distance between the plates of the condenser C; a pull (of say 200 grams) applied to the rod R decreased this distance by about the same amount. The elastic yield at the friction surfaces could thus be measured by the elastic yield of the standard rod R—the equality of the two strains being detected by the change in the frequency of the note heard in the telephones, as loads were lifted off and on. The rigidity of the supports was tested by applying pulls to various points other than the end of the rod R or the friction plate F.

Figure 3 shows the results obtained using as friction surfaces steel against cast iron. The elastic range of friction is clearly shown, and the "yield-point" on the graph when limiting friction was reached at the pull of 20 grams was fairly accurately determined by the manner in which the note heard in the telephones *slowly* altered after the pull was applied.

Fig. 3.



The effect produced by adding additional normal load to the friction plate F, such that the normal reaction between the surfaces was doubled (160 grams instead of 80), was also tested with the same two surfaces; and the experiments indicated that the elastic modulus for friction was thereby about doubled. For example, 10 grams pull on the friction plate now produced only about one-half the elastic yield previously observed: limiting friction, now reached with about 40 grams pull instead of 20, occurred at about the same total elastic yield. But it has to be confessed that it was found impossible to obtain conclusive results both as regards the elastic modulus and (to a less extent) as regards the position of the yield-point or of limiting friction. The surfaces, as

used, may be described as reasonably clean; they gave an ordinary coefficient of friction 0·25. They were not in the condition used, for example by Hardy * (which might perhaps be described as unreasonably clean) when a coefficient of limiting friction for steel on steel as high as 0·74 was obtained.

Feeling that the inconsistencies in the observations were in large measure due to a varying distribution of the friction grip and to varying cleanness of surface, it was decided to try further experiments using glass surfaces. By examining the layer between the surfaces under monochromatic (sodium) light and under white light, it was thought that the interference bands and colours seen should at least indicate the nature of the distribution of the friction grip.

Glass plates of good optical polish were obtained, smaller plates ($2\frac{1}{2}$ cm. square by 0·7 cm. thick) to rest on larger ones; and the friction apparatus illustrated by fig. 1 was modified so that a larger glass plate could be firmly bound to the base of the cast-iron support.

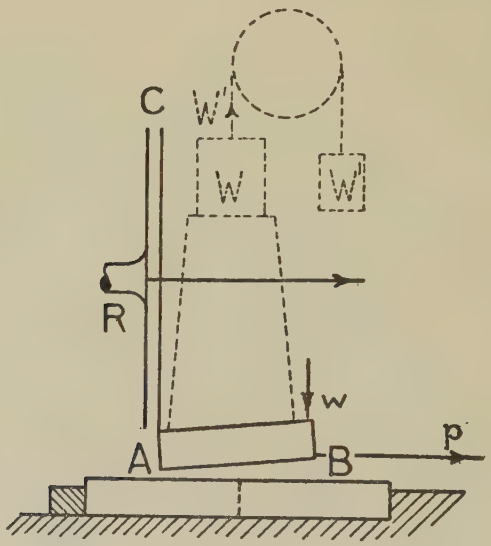
When the upper plate was laid on the lower one and the surfaces examined under monochromatic light, straight parallel equally spaced interference bands were observed, indicating that a wedge of air had been trapped between the plates (see fig. 4). A considerable load (W—W′) could be placed on the top plate without altering the number of bands seen, and the top plate could be slipped all over the lower one without appreciably altering the appearance of the interference bands. If white light were used no bands might be seen (the plates being then everywhere further apart than ·0002 cm.), or more usually coloured fringes † would be visible only at the narrow edge A. By manipulation of the top plate the parallel fringes could be replaced by a uniform colour, and by pressing the plates together this colour could be got to change and ultimately to pass through orange, straw, whitish, grey (when the film was of thickness about 10^{-5} cm.) to black; showing that the plates were of good optical polish and that whatever "dirt" had been trapped with the air between the plates could be pressed to very small thickness. The experiments were performed in an ordinary Glasgow atmosphere—not a very clean one—and it should be remembered that glass surfaces are exceptionally hygroscopic.

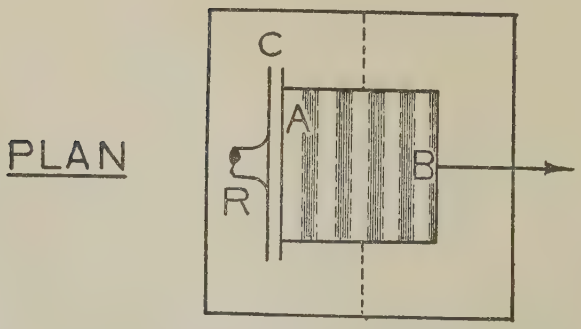
* 'Dictionary of Physics,' vol. i. p. 576.

† For a description of Newton's colours of thin plates and the thickness of film concerned see S. P. Thompson's 'Light Visible and Invisible,' p. 140.

What the nature and distribution of the forces (both normal and tangential) between the two friction plates may be, is a difficult question to answer. The layer between the plates may be of a thickness equal to many wave-lengths of

Fig. 4.





light, or, say, ten or more times the mean free path (10^{-5} cm.) of a gas molecule in the air. The particles in the solid surfaces are thus well beyond the range of direct action on each other—the range of molecular forces being of the order 10^{-6} cm. It is impossible to think of air layers of great thickness bound to the surfaces and (acting like rigid solids)

giving support to the upper plate. It might be thought that casual "dirt" in the atmosphere could give a prop-like support which would account for the wedge shape of the trapped layer, and on examining the layer with a microscope (using magnification of only 100) dirt specks could readily be detected; but the fact that the top plate could be moved over the lower one without appreciably altering the appearance of the interference bands is against this supposition. On the other hand, there is no doubt that the surface layers of water and of "grease," referred to by the late Lord Rayleigh in a paper "On the theory of Surface Forces"*, are of fundamental importance in friction problems. Such layers may be quite thick. For example, in an investigation of the heat transmitted through a boiler tube to clean and to oily water, decreases in transmission as great as 20 per cent. were observed, and attributed to oily layers which formed on the surface of the tube. The layers could be readily seen with the naked eye and could be rubbed off the surface of the tube at the conclusion of the experiments; they were estimated as being of thickness of the order one-hundredth of a millimetre.

The approximate permanence, when one plate is moved over the other, of the wedge shape of the trapped layer (illustrated in fig. 4) may be accounted for by regarding the upper plate to be at least partially supported cantilever fashion from the narrow edge A, where the forces in the film will be greatest. That such a cantilever support is possible may be demonstrated by allowing the upper plate to project far beyond the edge of the under one. If the plates at A are close enough to show straw and whitish colour, under white light, the upper plate can be permanently supported with its centre of gravity projecting well beyond the edge of the under plate.

Experiments were also performed using the ultramicrometer to illustrate the distribution of the supporting and tangential forces between two plates. For example, to test for the existence of forces at the wide edge B, of a trapped film, an under plate was cut in two (along the dotted line shown in fig. 4) and an attempt was made to slip out the portion under the wide edge B without permanently altering the capacity of the condenser C. The attempt failed. The half plate could be removed, but there was always a very small permanent change in the capacity of C, even when the half plate was moved very slowly so as to avoid a drag due

* 'Collected Papers,' vol. iii. p. 523

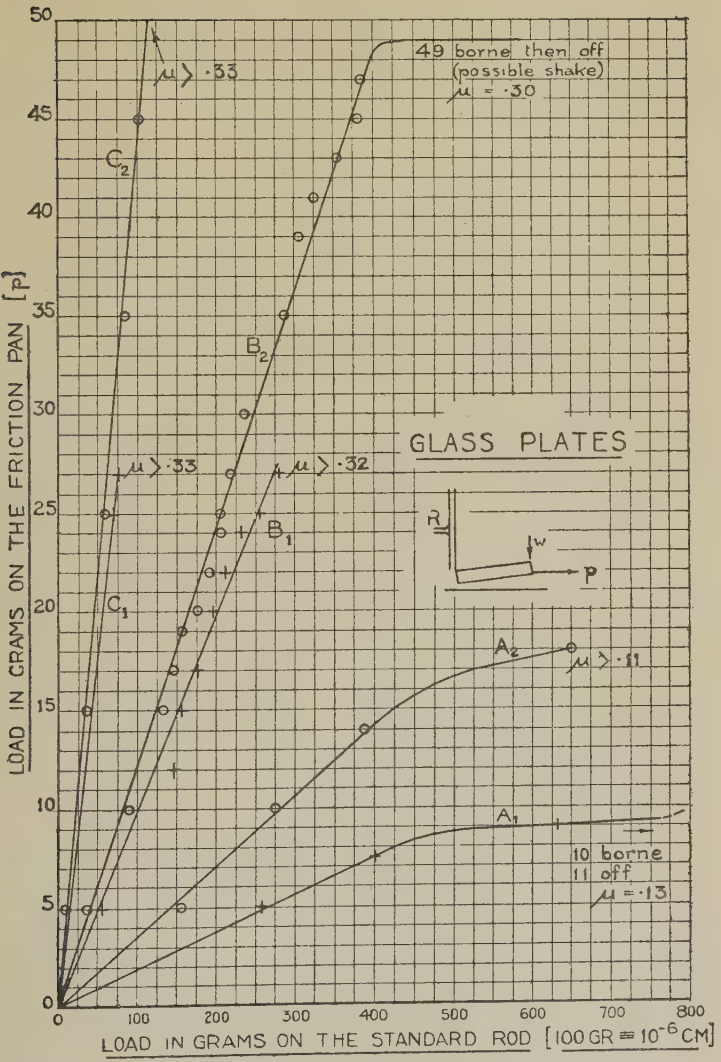
to viscosity in the layer. The small friction force thus shown to exist at B might be due to casual "dirt" or to unavoidable surface layers, but the main forces existed at the narrower edge A.

The effect on the capacity of the condenser C produced by adding a small weight w to the top surface of the upper plate was also investigated. By placing w above the wide edge B (fig. 4) a much greater elastic change occurred in the capacity of C than by placing w above the narrow edge A. An attempt was made to get more consistent results for the magnitude of the tangential strain produced by a pull p (see fig. 4), by applying a correction for this tilting action, a correction found by using the principle of moments and the magnitude of the tilt effect produced by w; but the attempt failed. For one reason, another complication had to be considered. The main grip at the edge A might be unsymmetrically distributed, so that, when the pull p was applied, elastic rotation might occur about a vertical axis somewhere about the letter A shown in the plan, fig. 4. Such a rotation might give rise to either an increase or a decrease in the capacity of the condenser C, depending on the position of the axis of rotation. It was several times observed that, when the friction plates were in very close contact, pull on the upper plate slightly increased the capacity of C instead of decreasing it; the increase in capacity due to the closer approach of the condenser plates at one edge (as the result of rotation) having been greater than the decrease in capacity due to separation at the other edge

When experiments on the elastic range of friction were started, it was thought that by comparing the elastic moduluses got for different materials with the ordinary elastic properties of the materials, some information might be obtained which would throw light on the question why it is we can speak of the coefficient of friction for glass on glass, brass on brass, steel on brass, and so on. The experiments just described show, however, that this idea had to be abandoned. One could never be sure that the distribution of the friction grip was the same in different experiments and the large corrections necessary for tilting and rotation effects would be unknown. Nevertheless the following results obtained with glass, brass, and steel surfaces all of good optical polish, and using two different normal loads, are thought to be worth recording.

Figure 5 shows results obtained with glass surfaces. The surfaces having been well cleaned, the top plate was gently

Fig. 5.



A_1, A_2—No colour with white light: 15 yellow bands with Na light.
w=1 gr. at edge produced same effect as 180 grs. on Rod R.
A_1—Normal reaction 83 grs.; A_2—Normal reaction 166 grs.

B_1, B_2—5 coloured fringes from edge A to rather more than half-way across the plate, then no colour to B, with white light.
B_1—Normal reaction 83 grs.; B_2—Normal reaction 166 grs.

C_1, C_2—Plate pressed on till whitish colour with white light—straw colour all gone
C_1—Normal reaction 83 grs.; C_2—Normal reaction 166 grs.

placed on the lower one and adjusted to give a narrow gap between the plates of the condenser C. A weight W, placed on a small stool (see fig. 4), was laid on the top plate—the normal reaction between the surfaces being then 166 grams, or on the average about 26 grams per sq. cm. The layer between the plates was then examined under white light and no coloured bands were seen, so that the plates were everywhere further apart than ·0002 cm. There were 15 bands seen with sodium light (the plates lying as illustrated in fig. 4), so, at the wide edge B, the plates were further apart than ·001 cm. Curves A_1 and A_2 were obtained together, a point on A_1, then a point on A_2. The reduction of the normal reaction (to half value) required for A_1 was obtained with as little disturbance of the friction plates as possible, by applying an upward pull W′ to the load W already placed on the top plate (see fig. 4). The curves indicate that the yield between the plates was not quite elastic right up to limiting friction ; and they indicate that there was a yield of more than ·001 cm. before slipping occurred, but of course, due to tilting, the condenser plates would not remain parallel. The effect produced (in the telephones) by placing a little additional load, $w=1$ gram, on the outer edge of the upper plate was the same as that due to applying 180 grams pull to the rod R. The coefficient of limiting friction was 0·13.

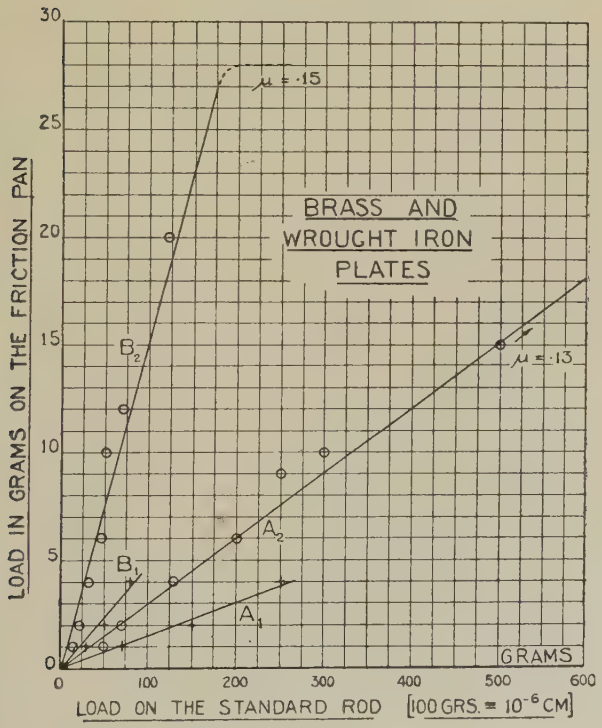
Curves B_1, B_2, were obtained with the glass plates in much closer "contact." Five coloured fringes were seen stretching more than half-way across the plates, so the layer between the plates varied in thickness from about ·00006 cm. to about ·00025 cm. The coefficient of friction was about 0·32.

For curves C_1, C_2, the top plate was pressed well on to the under one till the straw colour disappeared, leaving a whitish colour all over, so that the plates were separated by a distance of about ·000015 cm. Due to the breaking of a hook the pull p was not carried up to limiting friction, so the coefficient should probably be much greater than the value ·33 marked on the curves. With very clean surfaces values as high as 0·8, 0·9, greater than unity, had been obtained, but the handling of the plates necessary in these experiments rendered extreme cleanness impossible.

Figure 6 shows similar results obtained using brass and wrought-iron plates of good optical polish. The only method of investigating the character of the layer trapped between the plates was now by means of the little load w. In the case of both brass and of steel it was found that practically

the same slight tilting effect was produced by applying w anywhere round the edge of the top plate, and no tilting effect was observed when w was placed near the centre. This indicates that the surfaces in "contact" were slightly

Fig. 6.



A_1, A_2—Wrought iron plates.
A_1—Normal reaction 95 grs.
A_2—Normal reaction 176 grs.

B_1, B_2—Brass plates.
B_1—Normal reaction 99 grs.
B_2—Normal reaction 181 grs.

convex, so probably the under plates had been slightly buckled by the application of the strap which bound them to the massive cast-iron support.

The work described was carried out in the Natural Philosophy Department of the Royal Technical College, Glasgow, and the author desires to express his indebtedness to Messrs. Barr & Stroud Ltd. for kindly supplying the optically polished specimens, and to Professor James Muir, D.Sc., M.A., at whose suggestion the work was undertaken, for assistance rendered during the progress of the research.

Summary.

1. Reference is made to a paper on "Molecular Contact" by James S. Stevens, where it is shown, by an interferometer method, that in an ordinary experiment for finding coefficients of friction, elastic strains are produced before slipping occurs, of magnitudes one or two tenths of a wavelength of sodium light.

2. Apparatus is described for measuring this elastic range of friction by using Whiddington's oscillating-valve ultramicrometer method so that displacements of 10^{-7} cm. could be detected. Results are given for steel resting on cast iron, and an elastic range and "yield-point" for friction are clearly shown. The effect produced by doubling the normal load was tested. It was found that the elastic modulus for friction was thereby about doubled.

3. Inconsistencies in the observations led to experiments being performed with glass surfaces. The layer trapped between the plates could then be examined under monochromatic and under white light. The interference bands seen gave the shape and thickness of the trapped layer and indicated the distribution of the friction grip. This was also investigated by the ultramicrometer, *elastic* tilting and rotation effects, produced by small forces, being measured.

4. Curves are given showing the elastic range and yield-point of friction with glass plates (i.) separated by a trapped layer greater than ·001 cm. thick at the wide edge, (ii.) with a wedge layer ·00006 cm. to ·00025 cm., (iii.) with the plates pressed close together (about ·000015 cm. apart). In each case the effect of doubling the normal load was observed. Similar curves are shown for brass and wrought-iron plates.

LXXIV. *Latent Heats of Vaporization of Ethyl and Methyl Chlorides.* *By* G. W. C. Yates, *B.Sc.**

Method.

THE method used was one of electrical heating and consisted in measuring the amount of heat communicated to the calorimeter containing the liquid under investigation by the heating coil (Q) and equating this to: the heat lost by the evaporation of a determined weight of the liquid, the heat lost by emission (R), and the heat used in raising the temperature of the calorimeter during the experiment ($W\theta$). That is,

$$Q = ML + R + W\theta, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where W is the water equivalent of the calorimeter and contents, θ the rise in temperature from the beginning to the end of the experiment, M the mass of liquid evaporated, and L the latent heat of the liquid.

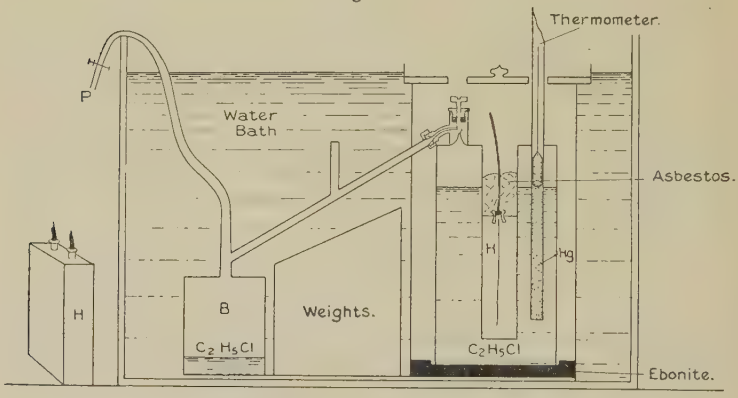
Description of apparatus used for Ethyl Chloride.

The copper calorimeter was entirely closed except for a cone tap, as shown in fig. 1. The temperature was recorded by a thermometer placed in a thick closed copper tube (communicating only with the exterior) filled with mercury, which ran most of the way down the inside of the calorimeter. The electrical heater (H) was placed in a rectangular prismatic opening in the calorimeter into which it just fitted. Good contact was ensured by the pressure of the liquid and vapour in the calorimeter. This arrangement also secured good stirring of the liquid by convection currents. Several types of electrical heater were tried (during the test experiments); the one ultimately used was of the shape shown in fig. 1. It had a resistance of about 50 ohms, and consisted of nickel-chrome wire embedded in allundum cement (an electrical insulator with good conductivity for heat) contained in a copper case. The vessel containing the calorimeter had a small condensing vessel B attached, and stood in a large bath as shown in fig. 1. The temperature of this bath was kept constant, in some cases by filling it with water and intermittently heating electrically so as to keep it at a constant temperature (above air temperature). The bath was stirred by hand. The lid of the vessel containing the calorimeter was of thick copper ($\frac{3}{8}$ in.) to secure it being as nearly as possible at the same temperature as the tank. The

* Communicated by Prof. C. H. Lees, D.Sc., F.R.S.

evaporated liquid could be condensed if desired in the vessel B. This was not always done, however, as the duration

Fig. 1.



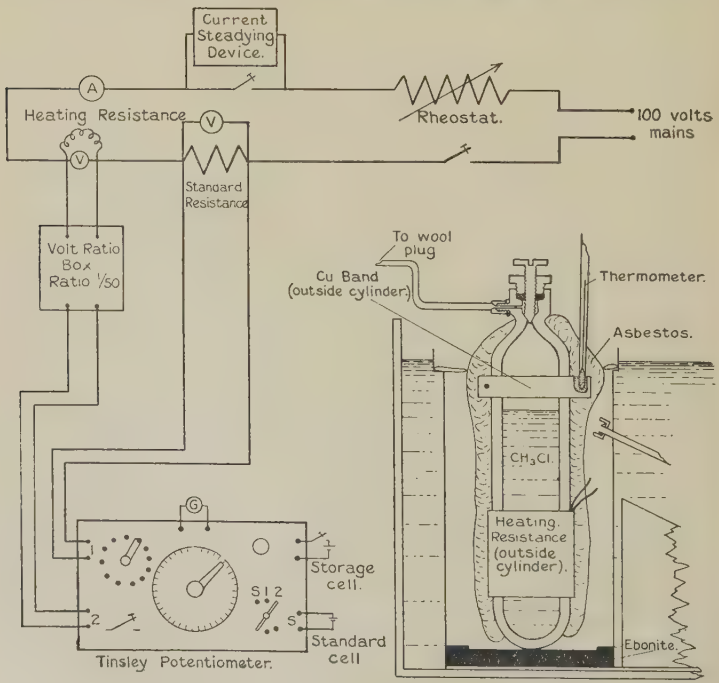


Fig. 2. Fig. 3.

of the experiment was thereby unduly lengthened. It was therefore allowed to escape at P.

The electrical circuit is shown in fig. 2. The input was determined by measuring the electrical pressures, across the heating resistance, and across a constructed standard resistance R, by means of a Tinsley precision potentiometer. The resistance of this coil R and attached voltmeter was measured by comparison with a standard ohm. The coil R was of No. 12 Eureka wire wound on a wooden frame 18 in. long and of cross-section 6 in. × 4 in. The spacing was very wide so that extremely little change of resistance would result from heating by the current.

Corrections were made for the shunt effect of the voltmeter and volt ratio box on the electrical heater. A current-steadying device was often used to diminish fluctuations in the voltage of the mains. A large coil through which the current passed produced a magnetic field which pulled on an iron bar attached to a pivoted beam which in turn raised a triangular copper electrode in a copper-sulphate resistance cell; an increase in current thus resulted in a simultaneous increase in circuit resistance, so that the increase in current would be much lessened. This device though useful was not entirely satisfactory.

Determination of R.

The heat lost by emission (R) may be determined in two ways.

(1) In which W is first determined.

Temperature time readings of the calorimeter are taken and plotted. A measured amount of heat is put in by the electrical heater (P) and the usual cooling curve drawn. The rise in temperature ϕ which would have obtained if there had been no loss of heat by emission is found.

Then $W\phi = P$, which gives W, the water equivalent. If $-\dot{\theta}$ is the rate of fall of temperature, at the mean temperature of the experiment (or mean rate of fall), $-W\dot{\theta}t$ will be the amount of heat lost by emission (R) during the experiment of duration t. $W\dot{\theta}$ can also be found as θ is known.

(2) From rate of rise of temperature with various heat supplies.

By measuring the rate of rise $\dot{\theta}_1$ of the calorimeter under a rate of heat supply (p) and the rate of rise $\dot{\theta}_2$ at the same temperature under no heat supply, we can determine the rate at which heat is gained or lost by emission. Considering

the calorimeter in the two cases (with and without electrical energy supply) between the temperatures ϕ and $(\phi+\Delta\phi)$, we have

$$p\,\Delta t_1 + \mathrm{E}\,\Delta t_1 = \mathrm{W}\Delta\phi$$

and $$\mathrm{E}\,\Delta t_2 = \mathrm{W}\Delta\phi,$$

or $$p + \mathrm{E} = \mathrm{W}\dot{\theta}_1$$

and $$\mathrm{E} = \mathrm{W}\dot{\theta}_2.$$

$$\therefore \quad p/\mathrm{E} = \mathrm{W}(\dot{\theta}_1 - \dot{\theta}_2)/\mathrm{W}\dot{\theta}_2,$$

or $$\mathrm{E} = \{\dot{\theta}_2/(\dot{\theta}_1 - \dot{\theta}_2)\}\,.\,p, \quad . \quad . \quad . \quad (2)$$

which gives E the rate of gain or loss of heat by emission between the temperatures ϕ and $(\phi+\Delta\phi)$. Then $-\mathrm{E}\,.\,t$ will be the amount of heat lost during the experiment of duration t.

$\mathrm{W}\theta$ can be written as $\mathrm{W}\dot{\theta}_2(\theta/\dot{\theta}_2) = \mathrm{E}(\theta/\dot{\theta}_2)$ and hence determined. E can be found at various temperatures by heating up the calorimeter over some considerable range and allowing it to fall over the same. The appropriate value for the mean temperature of the experiment can then be found.

Test Experiments.

Experiments to test the method were performed on water and carbon tetrachloride. The input was measured by means of ammeters and voltmeters subsequently standardized. In the later experiments on ethyl chloride a precision potentiometer was used for this purpose for the sake of greater accuracy. These experiments gave results in agreement to within $\frac{1}{2}$ to 1 per cent. (accuracy of the meters) of the accepted values.

The calorimeter used in these experiments was found to be unsatisfactory for ethyl chloride, so that shown in fig. 1 was substituted ; a better jacketing system also replacing that used in these experiments.

Typical Experiments.

I. C_2H_5Cl. Fri. 18/7/24.

From 1.50 P.M. to 2.26 P.M. the calorimeter fell in temperature owing to loss of heat by emission. The electrical heater was then put on and when a steady rate of rise was obtained the tap of the calorimeter was opened by a screwdriver (2.35 P.M.), allowing liquid to evaporate till 3.2 P.M., when the tap was closed. From 2.35 P.M. to 3.2 P.M. the temperature

remained fairly steady, the heat supplied by the electrical heater balancing that lost by evaporation and emission. After 3.4 the rate of rise of temperature again became steady. The experiment is taken from 2.34 P.M. to 3.4 P.M. (duration 30 mins.).

The calorimeter is raised 0°·75 C. during the experiments.
Mean temp. of experiment = 13°·8 C.
Mean rate of fall/min. due to emission = ·200° C./min. (at 13°·8 C.).
Loss of weight during experiment = 91·84 gms.

Then $(\dot{\theta}_1 + \dot{\theta}_2)W$ (see p. 819) = (·712 + ·167)W = mean rate of heat supply from 2.30 P.M. to 2.34 P.M.
Hence

$$W = 1523 \times \frac{10^7}{JS} \text{ cals.},$$

where J is the mechanical equivalent of heat and S the resistance of the standard (·9169 ohm).

Heat lost by emission, etc. during experiment

$$= (30 \times \cdot 200 + \cdot 75)W$$

$$= 10270 \times \frac{10^7}{JS} \text{ cals.}$$

Energy used up in evaporating the liquid

$$= (43440 - 10270) \times \frac{10^7}{JS} \text{ cals.}$$

Hence

$$L = \left\{ \left(33170 \times \frac{10^7}{JS} \right) \Big/ 91 \cdot 84 \right\} \text{ cals./gm.}$$

Then, as S = ·9169 ohms, and correcting for shunt effect voltmeter 3,300 ohms and volt ratio box 12,000 ohms, we have

$$L = 92 \cdot 5 \text{ cals. (at } 13 \cdot 8^\circ \text{ C.)}.$$

II. C_2H_5Cl. Mon. 1/12/24.

Here the water equivalent was determined by communicating a measured amount of heat to the calorimeter, observing the times and temperatures, and calculating the true rise by drawing the usual cooling curves. The energy used to counterbalance that lost by evaporation was put in for a certain portion only of the duration of the experiment.

This is more satisfactory than the method employed in the last experiment and was used oftener.

The temperature of the calorimeter dropped 0°·05 C. during the experiment.

Mean temp. of experiment = 13°·2 C,

Mean rate of fall/min. due to emission = 0°·0483 C./min. (at 13°·9 C.).

Loss of weight during experiment = 23·75 gms.

Duration of experiment = 28 mins.

Loss of heat during experiment due to emission = (28 × ·0483) W = 1·352 W.

Loss of heat due to emission − heat given out by calorimeter in falling 0°·05 C. = 1·302 W.

Corrected rise in temp. in subsidiary experiment for estimating water equivalent = 5°·05 C.

Then $$5{\cdot}05\ \mathrm{W} = 6434 \times \frac{10^7}{\mathrm{JS}},$$

$$\therefore \quad 1{\cdot}302\ \mathrm{W} = 1658 \times \frac{10^7}{\mathrm{JS}}.$$

Hence energy used in evaporation $= (10254 - 1658) \times \frac{10^7}{\mathrm{JS}}$

$= (8596) \times 10^7/\mathrm{JS}$ cals.

Then putting S = ·9169 ohm and correcting also for shunt effect of voltmeter and volt ratio box, gives

$$\mathrm{L} = 92{\cdot}6 \text{ cals. (at } 13°{\cdot}9 \text{ C.)}.$$

Modifications of apparatus for Methyl Chloride.

The method used for methyl chloride was the same as for ethyl chloride, but the cylinder was itself used as the calorimeter. The heating coil was wound round the lower portion of the cylinder and consisted of a thin layer of asbestos paper, a layer of resistance wire, with several layers of asbestos paper on top, and was held in position by a copper sheath. The temperature was read by means of a thermometer placed in a copper cup containing mercury, clamped tight to the cylinder by a copper band. The whole cylinder when in use was covered by asbestos lagging to prevent direct loss of heat from the coil by radiation and convection and in order that the thermometer might read the correct temperature more closely. The same containing or jacketing vessel was used as in the case of ethyl chloride. The expansion of the vapour on coming down to atmospheric pressure was arranged to occur at the far end of a long tube attached to the cylinder so that it should have no appreciable cooling effect on the cylinder. (See fig. 3.)

Results.

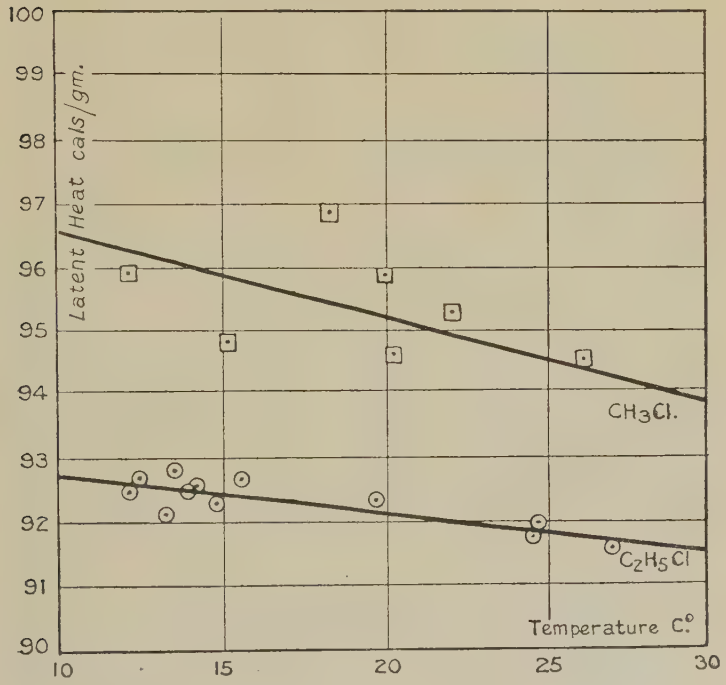
The results are given in the following two tables. They can be represented by

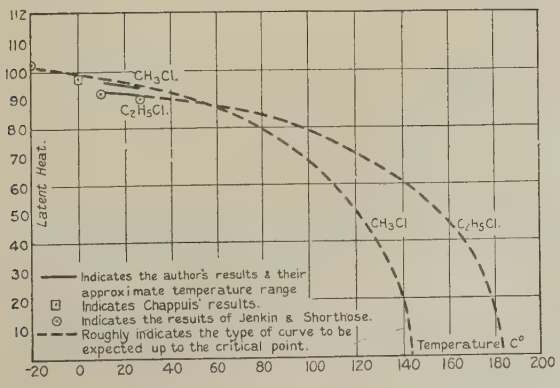
$$C_2H_5Cl \quad L_\theta = (93{\cdot}1 - 0{\cdot}06\,\theta) \text{ cals.}$$

$$CH_3Cl \quad L_\theta = (97{\cdot}7 - 0{\cdot}13\,\theta) \quad ,,$$

where L_θ is the latent heat at $\theta°$ C., and J is taken as $4{\cdot}200 \times 10^7$ ergs/calorie, over the range of temperature

Fig. 4.





Results of Experiments on Ethyl Chloride.

Mean temp. of exp.	Wt. of liquid vaporized.	Duration of exp.	$-\dot{\theta}$ in °C./min. (p. 819).	θ° C. (p. 819).	$(R+W\theta)$.	W.S.	$(R+W\theta)\times S$ in joules.	Input Q.S (joules).	$(Q-\overline{R+W\theta})\times S$.	L.
°C.	gms.	mins.								
12·1	19·22	21	0·023	−0·12	0·363 W	1413	513	7454	6941	92·5
12·2	118·27	34	0·177	−2·90	3·12 W	1250	3896	46726	42830	92·7
13·0	21·90	21	−0·013	−1·20	−1·47 W	1215	−1785	6090	7875	92·1
13·8	91·84	30	0·200	0·75	6·75 W	1523	10270	43440	33170	92·5
13·9	23·75	28	0·0483	−0·05	1·302 W	1274	1658	10254	8596	92·6
14·9	14·55	24	0·0159	0·31	0·692 W	1285	889	6132	5243	92·3
15·5	60·20	30	0·0716	−0·30	1·85 W	1004	1858	23660	21802	92·7
19·5	30·80	32	0·023	−0·13	0·606 W	1390	843	11943	11100	92·3
24·2	22·35	16	−0·090	−0·27	−1·71 W	1705	−2916	5095	8011	91·9
23·9	16·75	38	−0·015	−0·95	−1·52 W	1570	−2387	3616	6003	91·7
27·4	49·85	60	0·0172	−1·50	−0·47 W	1307	− 614	17180	17794	91·5

In deducing L from $(Q-\overline{R+W\theta})$ a correction of 1·6 per cent. is made for shunt effects; J is taken as $4{\cdot}190\times10^7$ ergs/calorie, and the standard resistance S has a resistance 0·9169 ohm.

Results of Experiments on Methyl Chloride.

Mean temp. of exp.	Wt. of liquid vaporized.	Duration of exp.	$-\dot{\theta}$ in °C./min. (p. 819).	θ° C. (p. 819).	$(R+W\theta)$.	W.S.	$(R+W\theta)\times S$ in joules.	Input Q.S (joules).	$(Q-\overline{R+W\theta})\times S$.	L.
°C.	gms.	mins.								
12·2	29·90	30	0·012	−0·50	−0·14 W	788	−110	11080	11190	95·9
15·0	8·55	40	0·000	0·077	0·077 W	809	62	3227	3165	94·9
18·2	24·00	30	0·020	−0·20	0·40 W	810	324	9397	9073	96·9
20·0	40·70	20	0·000	−0·10	−0·10 W	865	−86	15010	15096	94·9
20·2	26·80	26	−0·0154	−0·40	−0·80 W	939	−751	9160	9911	94·7
22·0	28·90	28	0·010	0·12	0·40 W	753	301	11050	10750	95·3
26·0	25·64	35	0·0973	−2·00	1·405 W	806	1133	10610	9477	94·7

In deducing L from $(Q-\overline{R+W\theta})$ a correction of 1·6 per cent. is made for shunt effects; J is taken as $4{\cdot}190\times10^7$ ergs/calorie, and the resistance of the standard resistance S is given its value 0·9169 ohm.

covered by these experiments (*i. e.* from about 10° C. to about 30° C.). These results together with those of Chappuis and Jenkin & Shorthose are shown graphically in fig. 4. The dotted curves up to the critical points are put in on the assumption that Thiesen's formula, $L_\theta = L_1(c-\theta)^{\frac{1}{3}}$, where L_θ represents the latent heat at θ° C., c the critical temperature, and L_1 a constant, holds for these liquids over this temperature range.

Conclusion.

The following is a list of previous determinations of the latent heat of ethyl chloride (in cals./gm.). The author's values are included for comparison.

Temp.	Regnault (1).	Regnault (2).	Jenkin & Shorthose*.	Yates.
12·5° C.	92·2	92·2	91·5	92·6
21·0° C.	90·7	91·9	89·8	92·0

Regnault determined the total heats †. The column Regnault (1) gives the latent heat calculated from these total heats using Regnault's specific heat values. The column Regnault (2) gives the latent heat calculated from these total heats and from Jenkin & Shorthose's specific heat values.

The following is a list of previous determinations of the latent heat of methyl chloride (in cals./gm.). The author's values are included for comparison.

Temp.	Observer.	Latent Heat.
0° C.	Chappuis ‡	96·9
−23°·8 C.	Shorthose §	102·45
+20° C.	Yates	95·2

The results of Jenkin & Shorthose were unpublished at the time the present investigations were commenced.

In conclusion, I wish to express my sincere thanks to Prof. Lees for suggesting this work, and for his most valuable advice throughout.

* Jenkin & Shorthose, Food Investigation Board (Dept. of Sci. and Indus. Research), Special Report No. 14 (1923).

† Regnault, *Ann. Chim. Phys.* (4) xxiv. pp. 423, 438 (1871).

‡ Chappuis, *Ann. Chim. Phys.* xv. pp. 498–517 (1888).

§ Shorthose, Food Investigation Board (Dept. of Sci. and Indus. Research), Special Report No. 19 (1923).

LXXV. *On the Parallel-Plate Condenser, and other Two-Dimensional Fields Specified by Elliptic Functions. By* W. B. Morton, *M.A., Queen's University, Belfast* *.

THIS note is a commentary on certain paragraphs in Sir J. J. Thomson's book, 'Recent Researches in Electricity and Magnetism,' published in 1893. The third chapter of that work is devoted to the discussion of a number of cases of the application of conjugate functions to electrostatics, first by use of Schwarz's transformation, and then by the indirect method in which assumed functional relations between $w=u+iv$ and $z=x+iy$ are interpreted in terms of electric fields. The cases

$$z=sn\,w,\ cn\,w,\ sn^{\frac{1}{n}}w,\ cn^{\frac{1}{n}}w,\ \log sn\,w,\ \log cn\,w,$$

are discussed in detail, and expressions are found for capacities and distributions of charge. At the end of the chapter the relation $z=Zw$ where Z is Jacobi's function defined by $Eamu-Eu/K$, is given to represent the case of two equal parallel strips forming the simplest kind of condenser. This is not developed further, owing to the want of tables of the function involved.

I have made drawings of all these fields, and in the last case I have supplied a detailed discussion on the lines of those given in the book for the other fields.

The calculations, from which the families of curves $u=$const. and $v=$const. were plotted, were straightforward and not laborious. The interval from 0 to K or 0 to K′ is divided into ten equal steps in each family, so that there are equal differences of potential between level surfaces and equal charges between the ends of lines of force. The modular angle is 30°, except in the last case, where it is 60°. It will be noticed that the interchange of the meanings of uv gives a different arrangement of conductors in the *sn* cases, but not in the *cn* cases. The former fact apparently was not noticed by Sir J. J. Thomson; he used *dn* to obtain the alternative case of *sn*. The functions *sn* and *dn* give identical diagrams.

* Communicated by the Author.

The Parallel-Plate Condenser.

The equation $z = \mathrm{Z}w$ when developed gives

$$x = \mathrm{Z}u + k^2 sn\,u\,cn\,u\,dn\,u\,sn^2v/(cn^2v + k^2sn^2u\,sn^2v)$$

$$y = -(\mathrm{Z}v + \pi v/2\mathrm{K}\mathrm{K}') + dn^2u\,sn\,v\,cn\,v\,dn\,v/(cn^2v + k^2sn^2us\,n^2v).$$

The elliptic functions of the quantity u are taken with modulus $k = \sin\alpha$, those of v with the complementary modulus $k' = \cos\alpha$. The origin of (xy) is at the centre of one of the plates, and the range u from 0 to K, v from 0 to K′ covers a quarter of the field. For when $v = 0$ we have $x = \mathrm{Z}u$, $y = 0$; and when $v = \mathrm{K}'$, $x = \mathrm{Z}u + cn\,u\,dn\,u/sn\,u$, $y = -\pi/2\mathrm{K}$. The function $\mathrm{Z}u$ vanishes for $u = 0$ and $u = \mathrm{K}$, and has a maximum for a certain value, u_1 say, determined by the equation $dn^2u_1 = \mathrm{E}/\mathrm{K}$. Therefore the equipotential $v = 0$ consists of the two faces of a strip of breadth $2\mathrm{Z}u_1$, with centre at the origin. The parallel line $v = \mathrm{K}'$ extends from the axis of y to infinity, and is the positive half of the line of symmetry midway between the two plates. Repeating the quarter-plane so obtained, we have the field of the condenser, the distance of the plates being π/K, breadth of a plate

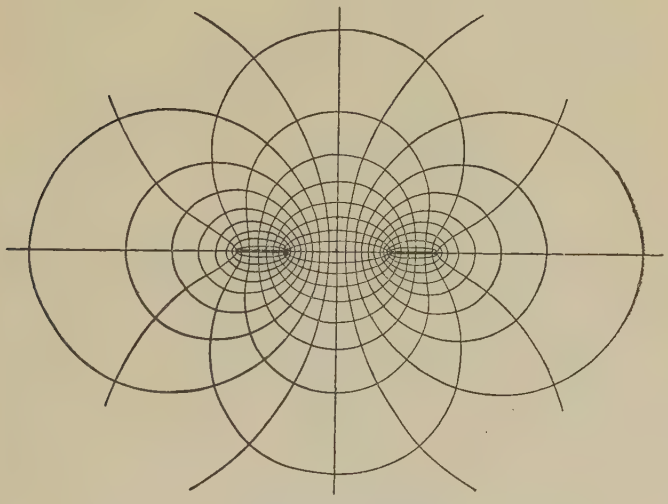
$$2\mathrm{Z}u_1 = 2\mathrm{Z}\{dn^{-1}\surd(\mathrm{E}/\mathrm{K})\},$$

charge per unit length on either strip $\mathrm{K}/2\pi$, difference of potential between the strips $2\mathrm{K}'$.

The capacity per unit length is thus $\mathrm{K}/4\pi\mathrm{K}'$. If calculated by the elementary formula, which holds when the plates are very close together, the value would be $\mathrm{K}.\,\mathrm{Z}u_1/2\pi^2$. The ratio of the actual capacity to this, $\pi/2\mathrm{K}'.\,\mathrm{Z}u_1$, may be called the "capacity-factor," while the ratio of spacing to breadth of plate $\pi/2\mathrm{K}.\,\mathrm{Z}u_1$ may be taken to specify the shape of the condenser. This ratio approaches zero as the modulus k approaches unity, for the maximum value of the function Z approaches one, while K becomes infinite in a logarithmic manner. The slowness with which K increases limits rather severely the application of the formulæ to practical cases; even when the modular angle is 89°, the ratio of distance to breadth is still ·459. The capacity-factor is plotted against the shape-constant on fig. 8, and the values of the modular angle are marked on the graph. The relation is almost linear in the range of the calculations, but the curve swerves round to touch the vertical axis at the limiting value unity. For $\mathrm{Z}u_1 = \mathrm{E}amu_1 - \mathrm{E}u_1/\mathrm{K}$, which approaches $(1 - u_1/\mathrm{K})$, so the capacity-factor approaches

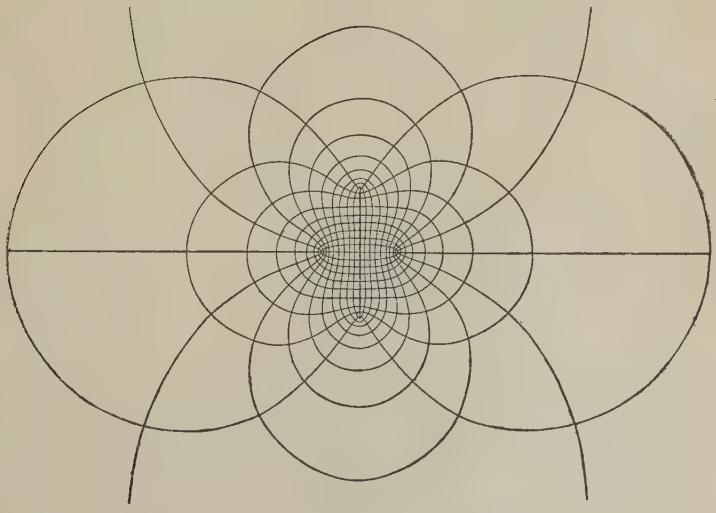
$(1+u_1/\mathrm{K})$, while the shape constant approaches $\pi/2\mathrm{K}$, and (capacity-factor $-$ 1) / (shape-constant) approaches $2u_1/\pi$. As k becomes equal to unity u_1 and K become infinite while the ratio u_1/K approaches zero.

Fig. 1.



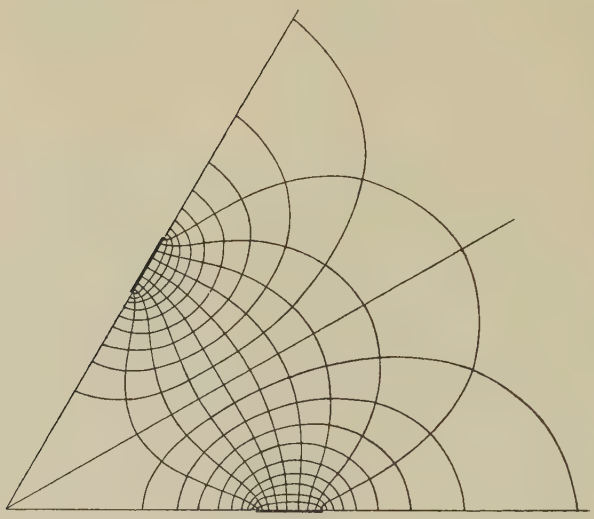
$z = sn\, w$ and $z = dn\, w$.

Fig. 2.



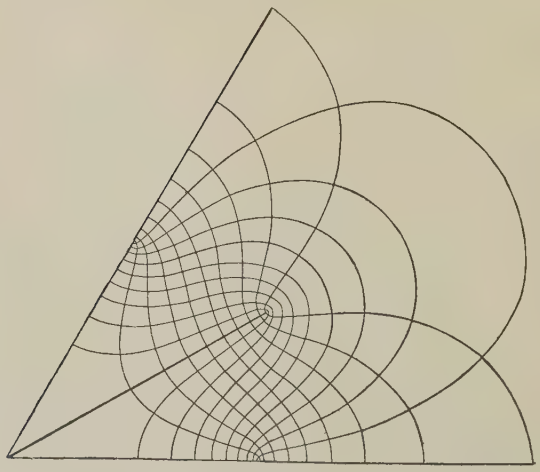
$z = cn\, w$.

Fig. 3.



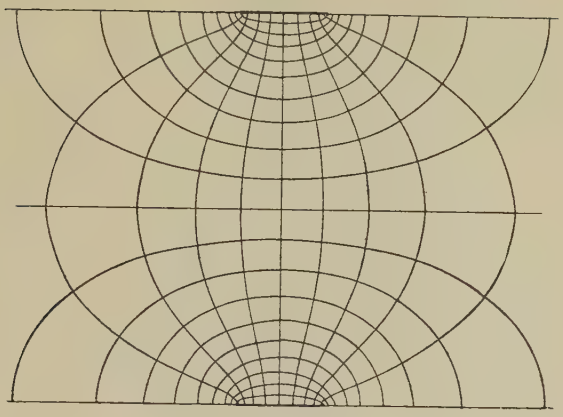
$z = sn^{\frac{1}{3}} w.$

Fig. 4.



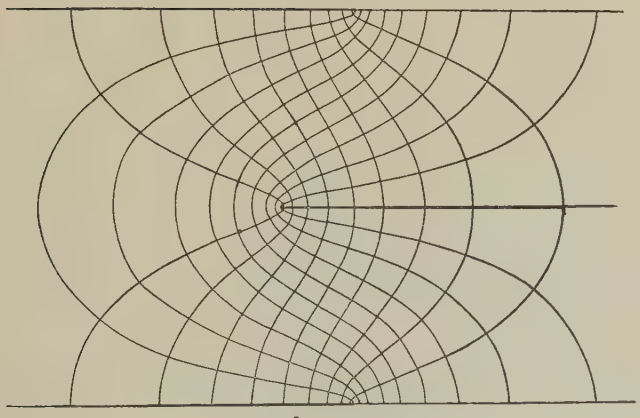
$z = cn^{\frac{1}{3}} w.$

Fig. 5.



$z = \log sn\, w$

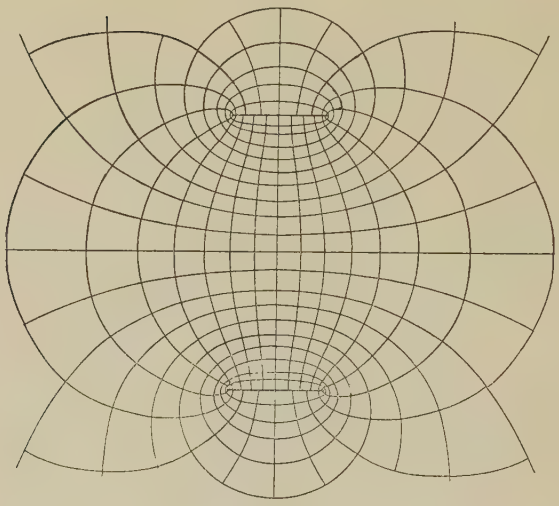
Fig. 6.



$z = \log cn\, w.$

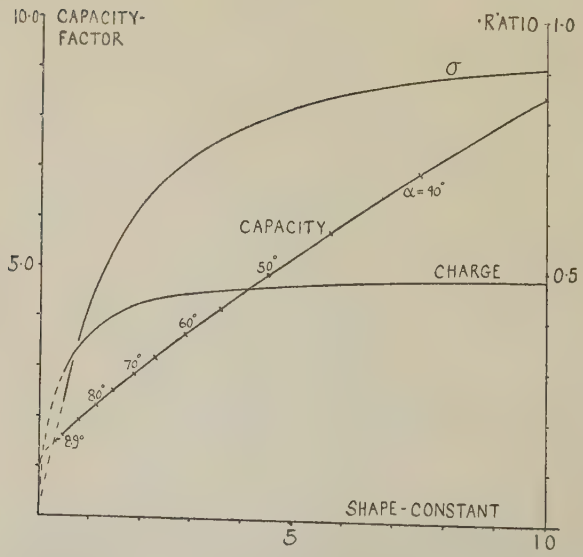
The other curves on fig. 8 show the variation, as the plates are drawn apart, of (1) the fraction of the whole charge on a plate which resides on the outer surface, and (2) the ratio of the surface-density at the centre of the plate on the outer side to that at the opposite point on the inner

Fig. 7.



$z = Zw.$

Fig. 8.



side. These are plotted on ten times the vertical scale of the other curve, the values being marked at the right-hand side of the diagram.

(1) The outer charge is to the whole charge as u_1 is to K. It follows that this curve, close to the origin, is a magnified copy of the graph for capacity-factor near the value unity. So for plates close together we have the result that the fractional increase of capacity, due to the curvature of the lines of force, is equal to the proportion of charge on the outside of the plates.

(2) We have $$4\pi\sigma = du/dx = du/dZu = 1/(dn^2u - E/K).$$

At the centre on the outer side $u=0$,
" " on the inner side $u=K$.

$$\therefore \quad \sigma \text{ outside}/\sigma \text{ inside} = \{(E/K) - k'^2\}/\{1-(E/K)\}.$$

The value ranges from 1 when $k=0$ to 0 when $k=1$.

This graph comes down to the origin at a finite angle with the horizontal. Near the limit $k=1$ we have

$$(\sigma\text{-ratio})/(\text{shape-constant}) = 2/\pi.$$

LXXVI. *The Stability of a Layer of Fluid heated below.* By Harold Jeffreys, *M.A., D.Sc., F.R.S.**

IT was shown by the late Lord Rayleigh † in 1916 that a liquid might be in stable equilibrium even if its density increased upwards, provided its viscosity was sufficiently great; and a detailed solution was obtained for the mode of decay or growth of disturbances in certain conditions. The results were in qualitative agreement with the experimental work of Bénard ‡, since repeated and extended by A. R. Low and Brunt §. The boundary conditions assumed by Rayleigh were somewhat artificial, and on account of the general interest of the question and the practical importance of some of its applications it is desirable to extend the work to allow for other possibilities.

The method adopted here is based on the principle of "exchange of stabilities." Let us consider for simplicity a system with a finite number of degrees of freedom. If we assume that in a free disturbance from the steady state all the departures vary with the time like $e^{\gamma t}$, where γ is a constant, γ will satisfy an algebraic equation with real coefficients. If then

* Communicated by the Author.
† Phil. Mag. xxxii. pp. 529–546 (1916).
‡ *Ann. d. Chimie et de Physique*, xxiii. p. 62 (1901).
§ 'Nature,' cxv. pp. 299–301 (1925).

the physical circumstances of the system vary continuously, all the roots of this equation will in general vary continuously. If the system is conservative, and passes from stability to instability, one of the admissible values of γ^2 changes sign by passing through zero. But the vanishing of a root of an algebraic equation implies that the constant term vanishes; and since γ is equivalent to $\partial/\partial t$, the departure from the steady state will satisfy a differential equation involving the independent variable only through its differential coefficients with regard to the time and not explicitly. Thus when the system is on the verge of becoming unstable the equations of motion will have a solution such that the departure from the steady state is different from zero and independent of the time. When instability is attained, the departure involved in this solution will be the one that will grow with the time.

The present system involves viscosity, and therefore is not conservative. It will still be true, however, that γ will involve an equation with real coefficients (possibly of infinite degree). If the rate of increase of temperature upwards is β (which will be taken as negative), it is fairly clear that when the kinematic viscosity ν is great enough any disturbance will die down exponentially with the time, so that every value of γ is real and negative. When ν diminishes algebraically, instability will arise when one of the values of γ first acquires a positive real part. If its imaginary part is also zero, the previous considerations will apply, and a steady departure from the state of equilibrium will be possible. But if as ν diminishes from ∞ to the critical value two of the values of γ coalesce and then become conjugate complexes, they may have finite imaginary parts when their real part vanishes, and the argument will break down. In this case the motion arising when instability develops will be an oscillation with gradually increasing amplitude. The impossibility of such a movement has not been proved in the present paper, but in Lord Rayleigh's discussion it was proved for a special type of boundary conditions, and here I shall only appeal to experiment to confirm that it is not the type of motion that actually does arise when equilibrium breaks down. I shall assume, then, that the uniformly stratified condition of the fluid first becomes unstable when the equations of the problem have a solution different from zero and independent of the time.

2. Let us assume that in the steady state the temperature is βz, the temperature on the floor being taken as zero; here

β is a constant and z the elevation above the floor. Take horizontal coordinates x and y; and let the components of velocity be u, v, and w. Let p be the pressure, V the temperature, ρ the density; and let suffix zero attached to any of them indicate its value at the same place in the undisturbed state, and an accent its departure from the undisturbed value. Let κ, ν, and α be the coefficients of thermometric conductivity, kinematic viscosity, and volume expansion. Then the equations of motion and of heat conduction are

$$\rho \frac{d}{dt}(u, v, w) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) p + \nu\rho\nabla^2(u, v, w) - (0, 0, g\rho), \quad . \quad . \quad . \quad (1)$$

$$\frac{d\mathrm{V}}{dt} = \kappa\nabla^2\mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the equation of continuity is

$$\frac{d\rho}{dt} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The equation of thermal expansion may be taken to be

$$\rho = \rho_c(1 - \alpha\mathrm{V}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ρ_c is the density at temperature zero; the fluid is supposed incompressible.

All these equations reduce to identities in the steady state except the equation of vertical motion, which gives

$$0 = -\frac{\partial p_0}{\partial z} - g\rho_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let us now neglect all squares of departures from the steady state. The d/dt reduces to $\partial/\partial t$ in (1), giving

$$\rho_0\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)(u, v, w) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) p' - (0, 0, g\rho'). \quad . \quad . \quad . \quad (6)$$

Forming the divergence of both sides of this equation, we may put ρ_c for ρ_0 when multiplied by a small quantity; then

$$\rho_c\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\nabla^2 p' - g\frac{\partial \rho'}{\partial z}. \quad (7)$$

From (3) (4) and (2)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho_c}\frac{d\rho}{dt} = \alpha\frac{d\mathrm{V}}{dt} = \alpha\kappa\nabla^2\mathrm{V}'. \quad . \quad (8)$$

Thus

$$\alpha\kappa\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2 \mathrm{V}' = -\frac{1}{\rho_c}\nabla^2 p' + g\alpha\frac{\partial \mathrm{V}'}{\partial z}. \quad . \quad (9)$$

Also from (2)

$$\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)\mathrm{V}' + \beta w = 0, \quad . \quad . \quad . \quad . \quad (10)$$

which determines w in terms of V′. Substituting in the equation of vertical motion we have

$$\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\mathrm{V}' = \frac{\beta}{\rho_c}\frac{\partial p'}{\partial z} - g\alpha\beta\mathrm{V}'. \quad (11)$$

Eliminating p' between (9) and (11),

$$\left\{\kappa\alpha\frac{\partial}{\partial z} + \frac{1}{\beta}\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)\right\}\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2\mathrm{V}' = -g\alpha\left(\frac{\partial^2 \mathrm{V}'}{\partial x^2} + \frac{\partial^2 \mathrm{V}'}{\partial y^2}\right). \quad (12)$$

If the depth is h, the ratio of the term involving $\kappa\alpha$ to that on the right is of order $(\nu\kappa/gh^3)$ which is in general very small. The term in $\kappa\alpha$ will therefore be neglected. (This approximation was made *ab initio* by Rayleigh.) At the same time we shall introduce the condition of marginal stability that $\partial \mathrm{V}/\partial t$ is zero. Then (12) simplifies to

$$\nabla^6 \mathrm{V}' = -\frac{g\alpha\beta}{\kappa\nu}\left(\frac{\partial^2 \mathrm{V}'}{\partial x^2} + \frac{\partial^2 \mathrm{V}'}{\partial y^2}\right). \quad . \quad . \quad . \quad (13)$$

Suppose now that V′ is proportional to $\sin lx \sin my$, the remaining factor being a function of z only. Put

$$(l^2+m^2)h^2 = a^2, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

so that a is a pure number. Put also

$$z = h\zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

so that ζ is zero at the bottom and unity at the top. Then (13) becomes

$$\left(\frac{\partial^2}{\partial\zeta^2}-a^2\right)^3 \mathrm{V}' = -\lambda a^2 \mathrm{V}', \quad . \quad . \quad . \quad . \quad (16)$$

where

$$\lambda = -\frac{g\alpha\beta h^4}{\kappa\nu}. \quad . \quad . \quad . \quad . \quad (17)$$

3. *The boundary conditions.*

At the upper and lower boundaries some condition must be satisfied by the temperature. If the fluid is in contact with a perfect conductor V′ will be zero; if there is no conduction across the boundary $\partial V'/\partial z$ will be zero.

Three conditions must also be satisfied by the velocity components. If there is a rigid boundary, u, v, and w must all vanish. By 2(10) w is proportional to $\nabla^2 V'$. With our hypothesis as to the form of V′ the conditions satisfied by u and v reduce to one. Since these components are zero at all points of the boundary, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is zero, whence by 2(3) $\partial w/\partial z$ is proportional to w, which is zero. Thus the conditions at a rigid boundary are that $\nabla^2 V'$ and $\frac{\partial}{\partial z}\nabla^2 V'$ are zero.

If in the undisturbed position of a free surface z is constant over it, in the disturbed state w for this value of z will be proportional to $\partial p'/\partial t$, which is zero for marginal stability. Thus $\nabla^2 V'$ is zero. Also $\partial u/\partial z$ and $\partial v/\partial z$ will be zero for all values of x and y over this surface. Hence by differentiating 2(8) with regard to z, we see that $\partial^2 w/\partial z^2$ is equal to $\frac{\partial}{\partial z}\alpha\kappa\nabla^2 V'$. Using the value of w from 2(10) we see that this implies that

$$\frac{\partial^2}{\partial z^2}\nabla^2 V' = \alpha\beta\frac{\partial}{\partial z}\nabla^2 V'.$$

The maximum values of the quantities involved in this equation, if not restricted to be surface values, would be in a ratio comparable with $1 : \alpha\beta h$. The latter quantity is the ratio, to the mean density, of the difference between the undisturbed densities at the top and bottom of the fluid, which we have already assumed to be small. Thus our condition will hold only if the quantity on the left, evaluated on the boundary, is only a small fraction of its maximum in the interior of the fluid. Only a trifling error will therefore be introduced if we replace the condition by the assertion that $\frac{\partial^2}{\partial z^2}\nabla^2 V'$ is zero at the boundary. Thus the conditions at a free surface are that $\nabla^2 V'$ and $\frac{\partial^2}{\partial z^2}\nabla^2 V'$ are zero.

Altogether three conditions must be satisfied at both the top and the bottom of the fluid, while the linear equation 2(16) is of the sixth order. Their consistency implies a relation

between the numbers a and λ; for a given a the lowest of the values of λ that satisfy this relation (if there are more than one) would give the value of β appropriate to marginal instability if this were the only value of a admitted. Then the value of a that gives the lowest λ so determined will give the first type of disturbance that will grow, and the associated λ will give the value of β for marginal instability.

4. *Rayleigh's problem.*

The conditions adopted by Rayleigh at the boundary are equivalent to the vanishing of V', $\partial^2 V'/\partial z^2$, and $\partial^4 V'/\partial z^4$. His solution therefore corresponds to a fluid with a free surface at both top and bottom, with the temperature maintained constant over both. This is not a natural system, but admits a simple solution, all the conditions being satisfied if V' is proportional to $\sin s\zeta$, where s is a constant, and $\sin s = 0$. Then 2(16) gives

$$(s^2+a^2)^3 = \lambda a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the lowest admissible value of λ is given by taking $s=\pi$. Then

$$\lambda = \frac{(\pi^2+a^2)^3}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We find that λ is least when

$$a = \frac{\pi}{\sqrt{2}} = 2{\cdot}22, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and then

$$\lambda = \frac{27}{4}\pi^4 = 651. \quad . \quad . \quad . \quad (4)$$

This result is equivalent to equations (42) and (44) of Rayleigh's paper.

5. *Solution by finite differences.*

The equation 2(16) could be formally solved in terms of exponential functions, but this method would be troublesome, since four of the coefficients of ζ in the indices would be complex, and the boundary conditions would lead to an equation for λ in the form of the vanishing of a determinant of the sixth order with many complex elements. The most convenient method seems to be that of L. F. Richardson, using central differences *. Let us denote by Z the factor

* Mathematical Gazette, July 1925.

of V′ that involves ζ. If the values of the dependent variable are given at equal intervals d of the independent variable, thus,

$$Z_{-3}, Z_{-2}, Z_{-1}, Z_0, Z_1, Z_2, Z_3, \ldots\ldots,$$

the basis of the method is to replace differential coefficients by finite centred differences: thus for example those centred at $\zeta=0$ are

$$Z' = \frac{1}{2d}(Z_1 - Z_{-1}),$$

$$Z'' = \frac{1}{d^2}(Z_1 - 2Z_0 + Z_{-1}),$$

$$Z''' = \frac{1}{2d^3}(Z_2 - 2Z_1 + 2Z_{-1} - Z_{-2}),$$

$$Z^{iv} = \frac{1}{d^4}(Z_2 - 4Z_1 + 6Z_0 - 4Z_{-1} + Z_{-2}),$$

$$Z^{vi} = \frac{1}{d^6}(Z_3 - 6Z_2 + 15Z_1 - 20Z_0 + 15Z_{-1} - 6Z_{-2} + Z_{-3}).$$

If the boundary condition at $\zeta=0$ is that Z, Z″, and Z‴ vanish, these show that the neighbouring values of Z are of the form

$$Z_2 - 4Z_1, \ -Z_1, \ 0, \ Z_1, \ Z_2, \ \ldots .$$

This is the condition for a conducting rigid boundary. For a non-conducting free surface Z′, Z″, and Z^{iv} vanish, and we must have the values

$$2Z_0 - Z_2, \ Z_0, \ Z_0, \ Z_0, \ Z_2, \ \ldots .$$

The differential equation being of the sixth order, we must substitute for the derivatives from our central-difference formulæ at all points far enough from both ends for the sixth derivative to be formed by this process. The resulting equations should determine λ, and then the approximation may be improved by repeating the calculation for a larger number of intervals and using the principle that the error is of the form $Ad^2 + Bd^4$.

6. *Case of two rigid conducting boundaries.*

The simplest distribution of values of the independent variable to take consists in assuming a value of Z at $\zeta=\frac{1}{2}$, say b, and allowing all others to be determined by the

boundary conditions. Then we have $d=\frac{1}{2}$, and

ζ	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
Z	$-4b$	$-b$	0	b	0	$-b$	$-4b$

Forming the difference equation for the middle value, we find

$$\lambda a^2 = 1024+192a^2+24a^4+a^6. \quad . \quad . \quad . \quad (1)$$

To this approximation the minimum value of λ is 540, and occurs when $a=2{\cdot}33$.

Next consider the values of Z at intervals of $\frac{1}{3}$. The solution must be either symmetrical or antisymmetrical with regard to the median plane, and it is clear that the symmetrical solution will be the first to give instability, for it implies that the columns of warm and cold fluid extend throughout the whole depth. With these data only one value of Z needs to be assigned arbitrarily, and we have

ζ	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{3}$	$1\frac{2}{3}$
Z	$-3b$	$-b$	0	b	b	0	$-b$	$-3b$

The difference equation at either $\frac{1}{3}$ or $\frac{2}{3}$ gives

$$\lambda a^2 = 2187+243a^2+27a^4+a^6, \quad . \quad . \quad . \quad (2)$$

whence the minimum value of λ is 793, and occurs when $a=2{\cdot}69$.

Considering intervals of $\frac{1}{4}$, we now have two values of Z not fixed already, and we take

ζ	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$
Z	$-8b+c$	$-3b$	0	$3b$	$4b+c$	$3b$	0	$-3b$	$-8b+c$

Two independent difference equations must hold at $\frac{1}{4}$ and $\frac{1}{2}$; they lead to two equations homogeneous in b and c. By elimination we have, after some reduction, the quadratic for λ,

$$\{10922\tfrac{2}{3}+(512-\lambda)a^2+32a^4+a^6\}$$
$$\times\{169301\tfrac{1}{3}+(8704-\lambda)a^2+160a^4+a^6\}$$
$$=(21845\tfrac{1}{3}+1024a^2+16a^4)(60074\tfrac{2}{3}+2048a^2+32a^4). \quad (3)$$

From the two examples already worked out, assuming that the error in the value of a that makes λ a minimum is proportional to d^2, we see that the true value of a should be nearly 3. Substituting this in (3) and solving for λ, we find that $\lambda=928$. The error of λ is proportional to the square of the small error of a, and therefore is probably

unimportant. Thus we have found that when d takes the values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, the minimum values of λ found are respec- 540, 793, and 928. Extrapolating on the hypothesis that the error is of the form Ad^2+Bd^4, we find that the true value of λ is about 1140.

A preliminary solution was constructed by forming the simplest polynomial that would satisfy the boundary conditions and then finding how far it failed to satisfy the differential equation. Other terms were then added to improve the approximation, which was fairly rapid: λ was found to be about 1200. The initial computations were, however, very laborious, and on the whole Richardson's method is much more satisfactory.

7. *Rigid conducting boundary at base: free non-conducting surface at top.*

The method of the last section applies, except that there will no longer be symmetry about the median planes. The first approximation assumes the values

ζ	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
Z	$-3b$	$-b$	0	b	b	b	$2b$

The equation for λ turns out to be

$$\lambda a^2 = 384 + 192a^2 + 12a^4 + a^6, \quad . \quad . \quad . \quad (1)$$

and the appropriate values are $a=2{\cdot}08$, $\lambda=256$.

In the second approximation we take

$\zeta=$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{3}$	$1\frac{2}{3}$
Z=	$c-4b$	$-b$	0	b	c	c	c	$2c-b$

Writing $3f$ for a, and 81μ for λ, we have the equation for μ,

$$\{3+3f^2+3f^4+2(f^6-f^2\mu)\}\{39+5f^2+15f^4-2(f^6-f^2\mu)\} = -(5-3f^2-3f^4)(11+9f^2+3f^4) \quad . \quad . \quad (2)$$

Substituting for f^2 in this the values 1·000, 0·885, 0·770 (the intermediate one being chosen because it makes the right side vanish), we find that μ takes the values 5·34, 5·31, 5·39. By interpolation the minimum of μ is 5·30, occurring when $f^2=0\ 911$. Thus $a=2{\cdot}86$ and $\lambda=430$.

From the results obtained for $d=\frac{1}{2}$ and $d=\frac{1}{3}$ we may now extrapolate on the supposition that the error is proportional to d^2. We find that $a=3{\cdot}49$, $\lambda=571$.

As might be expected from the fact that variations of

temperature over the upper surface are now possible, instability sets in for a smaller temperature gradient than in either of the cases already considered. But at the same time the absence of the conducting upper surface delays the spread of heat horizontally ; it is presumably to this that the higher value of a, corresponding to a shorter critical wave-length, is to be attributed.

8. *Non-conducting rigid boundaries at top and bottom.*

The boundary conditions in this case are that Z', Z'', and Z''' vanish at both boundaries. Returning now to the equation 2(16), which takes the form

$$\left(\frac{d^2}{d\zeta^2}-a^2\right)^3 Z = -\lambda a^2 Z,$$

we see that the equation and all the boundary conditions are satisfied by a constant value of Z, provided that

$$\lambda = a^4.$$

Hence instability arises first for a zero, that is, for infinite wave-length, and then for λ very small. In this case therefore there is no possibility of complete stability with the higher temperature below. The movement once initiated cannot be maintained, of course, because there is no means of supplying new energy to the fluid ; if the hotter fluid is originally underneath, it will exchange places with that above, and the motion will then gradually die down.

The same would apply if the upper surface was free, so long as it is non-conducting. Conduction over either boundary excludes the possibility of variations of the temperature there, and solutions of the type just considered do not arise then ; a conducting boundary is essential to the stability of the arrangement with the hotter fluid below.

9. The conditions of greatest physical interest are probably those discussed in 6 and 7. These are the ordinary cases of liquid heated below, the upper boundary being in the one case a conducting solid, and in the other a non-conducting free surface. I was led to the former by the suggestion of L. H. Adams * that viscosity would play an important part in determining the vertical distribution of temperature in the earth's crust just after the solidification of the exterior ; for a high viscosity would tend to damp down eddy currents, and the underlying magma might therefore cool at all points

* J. Wash. Acad. Sci. xiv. pp. 459–472 (1924).

until the viscosity everywhere had reached the critical value needed to prevent such currents, so that the distribution of temperature would become such as to make the viscosity nearly independent of depth. Let us try, then, to estimate the viscosity needed. For a rock magma the thermometric conductivity is probably about 0·01 c.g.s.; α is about 2×10^{-5}. Let us find the value of ν needed to prevent eddy currents in a layer 100 km. thick with a temperature difference of 100° between the top and bottom, so that $\beta=-10^{-5}$. Then from 2(17)

$$\lambda = 2\times10^{23}/\nu,$$

while the critical value of λ is about 1140. Thus ν must be about 2×10^{20} cm.²/sec. to prevent the development of eddy currents that would redistribute the temperature. This is far above any viscosity yet measured, and a fluid with such a viscosity would be experimentally indistinguishable from a solid. For the same value of β the critical value of ν varies as h^4, so that it would be very much reduced if we considered a smaller depth of fluid. If the substance acquires even a small permanent strength at these high viscosities, eddies would be prevented entirely, so that the temperature distribution when currents ceased would be determined by the melting-point. A further tendency to stability would be introduced if there was a concentration of heavier materials towards the bottom. On the whole, then, it seems likely that viscosity would not be the determining factor, though the present discussion does not close the question.

10. The non-conducting free surface above represents the ordinary heating of a fluid from below. There is a familiar domestic instance of complete stability in this case, namely, the burning of porridge when it is not kept stirred. The high viscosity of the material can prevent currents even when the free surface is at ordinary temperatures and the bottom is charring. Taking h equal to 3 cm., βh equal to $-400°$, κ and α equal to the corresponding quantities for water, say 0·0015 and 10^{-3} (both varying notably with temperature), we have

$$\lambda = 10^7/\nu,$$

while the critical value of λ is about 571. Thus stability would be maintained if ν were something like 1700 cm.²/sec. The viscosity of Lyle's Golden Syrup is given by Kaye and Laby (1919) as 1400 c.g.s., so that 1700 for porridge at the stage when it burns is not unreasonable. (It is capable of

being poured, so that we are dealing with a case of viscous stability and not of solidification.)

Of course the effect of stirring is to promote rapid mixing and thereby to reduce the difference of temperature between the top and the bottom, the supply of heat at the bottom remaining the same. The eddy currents introduced by stirring take the place of those caused purely by heating, so that the heat can be got rid of at a temperature gradient much less than that needed for instability due to heating alone.

LXXVII. *Growth of Eddies in a Viscous Fluid.*

By Prof. H. Levy, *M.A., D.Sc., Imperial College of Science* *.

THE mechanism of eddy production in a viscous fluid is not yet precisely understood, although as an observed phenomenon it is among the most common. Little more can be said to be known about it than that it appears almost invariably to be associated with states of unstable flow and with turbulent motion generally.

The works of Osborne Reynolds and that of a large number of subsequent investigators, including many of those engaged in aerodynamical research, point to the fact that for many conditions of viscous flow there would appear to exist a critical state corresponding to a particular value of the number vl/ν for which the motion passes from one of comparative steadiness and apparent stability to one of instability and possibly turbulent and eddying motion. Both below and above this critical phase, vorticity in the mathematical sense is known to exist in the fluid, but it is generally true that only above the critical position does the spinning or eddying motion which is naturally associated with vorticity make itself apparent to any marked extent. What in fact is an eddy? and if as would appear reasonable it is nothing but a vortex grown to visibility, how does a vortex *grow* in a viscous fluid?

Consider what happens in the neighbourhood of the sharp projecting edge of a body in motion in a fluid. At slow speeds—that is, below the critical value of the appropriate Reynolds' number—a state of affairs closely resembling the classical "discontinuous" motion appears to exist for a short distance behind the body. The surface

* Communicated by the Author.

of "discontinuity," or rather the surface of most rapid velocity gradient, starts off from the projecting edge, for a short distance remains clearly defined, and then becoming sinuous in form loses its identity in the general stream of fluid. At higher values of Reynolds' number the sinuosities increase in magnitude close to the projecting edge, and the discontinuous surface appears to wind itself up at points forming nuclei round which more and more fluid circulates.

Maturity for the eddy so formed appears to be attained when in the process of its passage down stream its distance from the projecting edge becomes too great for the vorticity there being produced to be drawn within the ambit of the receding eddy, and a new centre of rotation begins to make its appearance. The subsequent history of the eddy already shed appears merely to be one of gradual decay as its energy is absorbed by the viscous action of the fluid.

The breaking up of the surface of discontinuity into distinct eddies is not easily explained by an examination of the analogous process in a perfect fluid. Kelvin has indeed shown that such a surface is unstable, but it does not follow that anything further will occur in that case than a mere drastic change of shape of the surface. The separation up into *discrete* vortices does not appear to be necessarily involved.

Moreover, this vital distinction does exist between the interaction of separate vortices in a perfect fluid and that of eddies in a viscous medium, that whereas in the former a system of vortices during their interplay will always retain their separate identities irrespective of their relative strengths and signs, in the latter case experiment shows that an eddy may change in intensity apparently by directly absorbing any number of others, the system coalescing into one or more distinct eddies, and then gradually decaying.

If the foregoing description is accurate in its general outline, an essential feature is that the interaction between vortices of the same sign in a viscous fluid should be such as to lead towards coalescence particularly in the early stages of formation, that is to say when the motions are slow. This, then, is a problem in viscous fluid that might legitimately be studied on the assumption that the inertia terms in the equation are negligible.

Taking the simple case in which each element of the fluid is rotating in a plane parallel to (xy) about the axis of z, it can easily be shown that, if ψ be the stream function

and $r^2=x^2+y^2$, the equation of motion is

$$\frac{1}{\nu}\frac{\partial\psi}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

while the pressure equation gives

$$p-p_0=\int_{\infty}^{r}\frac{\rho\Omega^2}{r}\,dr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where Ω is the tangential velocity at any point. These equations may be established directly from a consideration of the motion of a single element or by direct integration of the usual equations in (xy) coordinates along a radius and round a circle with centre at the origin.

Any solution of (1) will correspond to a maintained or a decaying rotatory motion about the z axis. Any given initial motion of this type established at $t=0$ may be examined for its subsequent history by considering the solutions of (1) of the type

$$\sum_{1}^{\infty} e^{-\lambda_n t} f_n(r),$$

where it is easily seen that $f_n(r)$ is the Bessel Function of order zero of a certain multiple of r. Let us instead seek for solutions which are functions of $r^2/t=\theta$.

Then

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial\theta}\left(-\frac{r^2}{t^2}\right)=-\frac{\theta}{t}\frac{\partial}{\partial\theta},$$

$$r\frac{\partial}{\partial r}=\frac{\partial}{\partial\theta}\left(\frac{2r^2}{t}\right)=2\theta\frac{\partial}{\partial\theta},$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)=\frac{4}{r^2}\theta\frac{\partial}{\partial\theta}\left(\theta\frac{\partial}{\partial\theta}\right).$$

Thus (1) becomes

$$-\frac{\theta}{\nu t}\frac{\partial\psi}{\partial\theta}=\frac{4\theta}{r^2}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial\psi}{\partial\theta}\right)$$

or

$$4\frac{d}{d\theta}\left(\theta\frac{d\psi}{d\theta}\right)+\frac{\theta}{\nu}\frac{d\psi}{d\theta}=0,$$

$$\theta\frac{d\psi}{d\theta}e^{\theta/4\nu}=\mathrm{A},$$

Thus

$$\psi=\mathrm{A}\int\frac{d\theta\,.\,e^{-\theta/4\nu}}{\theta}+\mathrm{B}.$$

It follows that

$$\psi = \mathrm{A} \cdot \int_0^r \frac{dr(1-e^{-r^2/4\nu t})}{r} = \mathrm{A} \int_0^\phi \frac{d\phi(1-e^{-\phi})}{\phi}$$

is a solution of (1) where $\phi = r^2/4\nu t$.

Nature of the Solution.

When $t = +0$, the distribution is that due to a single line vortex of the logarithmic type coincident with the Z axis. As $t \to +\infty$, $\psi \to 0$ and $\frac{\partial \psi}{\partial r} \to 0$, so that the motion represents that of the decay of this logarithmic vortex in a viscous fluid.

The vortex will be initially of the strength K if $\mathrm{A} = \frac{\mathrm{K}}{2\pi}$. For circles and times for which $r^2/4\nu t$ is constant the circulation will be constant and equal to $\mathrm{K}(1-e^{-r^2/4\nu t})$, so that a circle (r_0, t_0) will continue to enclose a constant amount of vorticity if its radius grows at a rate $\frac{r_0^2}{2t_0} \cdot \frac{1}{r}$, or alternatively if the *area increases at a constant rate.*

At all times subsequent to $t=0$ the velocity at the centre is zero, while the angular velocity there is a maximum and dies off as t^{-1}.

The circle of maximum velocity is given by $r^2/4\nu t = \lambda$, where $e^\lambda = 1 + 2\lambda$, *i. e.* $\lambda = 1{\cdot}257$.

The solution so far obtained satisfies the equations of motion without restriction as regards slow motions. If now we consider the problem initially proposed—that of the interaction of eddies—we may legitimately impose the restriction regarding slowness of motion. In this case, the equations of motion become linear in form, viz.:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} - \nu \nabla^2 u,$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial v}{\partial t} - \nu \nabla^2 v,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

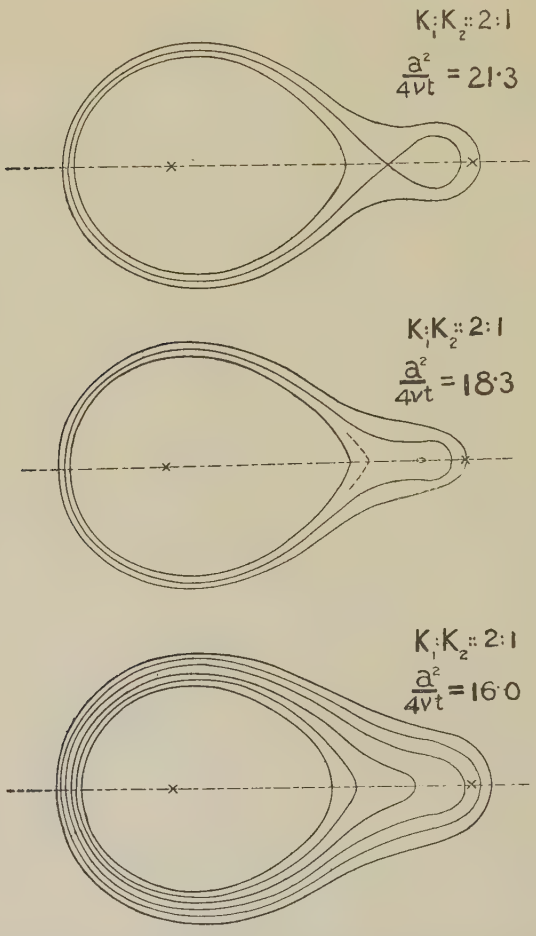
Solutions of the form already found may be superposed, say:

$$\psi_1 = \frac{\mathrm{K}_1}{2\pi}\int_0^{\phi_1} \frac{d\phi_1(1-e^{-\phi_1})}{\phi_1}, \quad \psi^2 = \frac{\mathrm{K}_2}{2\pi}\int_0^{\phi_2} \frac{d\phi_2(1-e^{-\phi_2})}{\phi_2},$$

where

$$\phi_1 = \frac{r_1^2}{4\nu t}, \quad \phi_2 = \frac{r_2^2}{4\nu t},$$

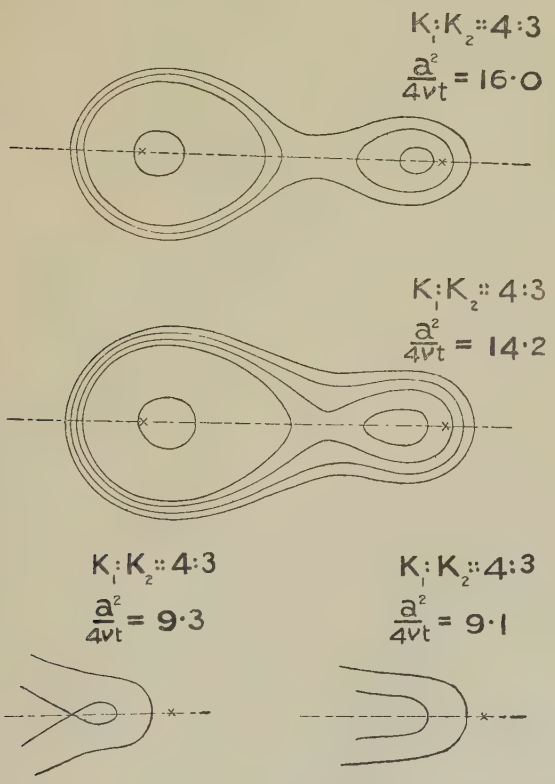
and where r_1 and r_2 are the distances of any point from the initial centres of the eddies. The stream function ψ at any point will be $\psi_1+\psi_2$, and the motion will be that arising from the sum of the separate effects of the eddies.



The geometrical shape of the stream lines at any instant will depend on the values assumed by $a^2/4\nu t$ and K_1/K_2, where a is the distance between the eddies at birth and K_1, K_2 are their initial strengths.

In the attached figures the stream lines are drawn for different values of K_1/K_2 (taken positive) and $a^2/4\nu t$. Thus they represent the interaction between two eddies of like

but unequal strength created at the same instant. It is clear that with given $a^2/4\nu t$ the stream lines for K_2/K_1 will merely be the reverse of those for K_1/K_2. As t increases, the eddies rotate round each other and the two centres of rotation as long as they remain distinct move towards each other. They do not, however, remain distinct up to coalescence, since for each value of K_1/K_2 there is a



particular time, or more properly a particular value of $a^2/4\nu t$ at which the closed curves round the weaker eddy vanish and for smaller values of $a^2/4\nu t$, that is for subsequent times, the stream lines consist merely of a single system of closed curves. This critical value determines the total life of the weaker eddy as a distinct entity. It is of course a function of K_1/K_2, and could be determined analytically from the expression for ψ if required, instead of graphically as here.

This analysis indicates, then, how eddies grow by absorbing their weaker neighbours while at the same time their combined energies degrade gradually into heat.

Now consider what happens in the neighbourhood of a cylinder past which a fluid is in motion. It is known that in the immediate vicinity of the surface there is a layer of fluid of the order of thickness $10^{-2} \times l$ where l is the length of the body, of exceedingly high velocity gradient ranging from zero velocity at the surface of the body to almost the full value V on the outer surface of this layer. Let us suppose that along the outer layer a series of elementary vortices are created spaced on the average a distance a apart. The strength from vortex to vortex will vary, according to its position along the surface and to the speed of the fluid.

Neighbouring vortices interacting as they pass along the surface will coalesce, and if the time taken for the vortices to pass from nose to tail of the body is large compared with the time of total coalescence, two large eddies, one on the upper surface and one on the lower, of opposite signs will be discharged at the tail. Meanwhile the process of absorption will have recommenced near the nose, a new eddy will grow like a snowball and be finally discharged: thus a periodic generation of eddies will result.

Now let S be the length of surface from the stagnation point at the nose, to the point of departure of the final eddy. If V is the velocity of the fluid at infinity the velocity of the fluid on the outer surface of the boundary layer will range from zero at the nose to approximately V at the point of departure, and the time taken by the accumulated eddy to pass over this distance is approximately $2S/V$. If, however, the velocity of the fluid be increased this time will be diminished, so that the interval during which eddy coalescence will occur along the surface of the body will diminish. This suggests at once that if the present standpoint is accurate there may exist a critical speed above which periodicity in eddy formation should disappear. It is not difficult to estimate the order of Vl/ν at which this ought to occur. The criterion will be found by recognizing that if the time of passage of a particle of fluid from stagnation point to leaving point is less than the time required to effect concentration between a pair of vortices, no very definite periodicity will be evident, since the vorticity will not make itself evident as eddying.

We will assume that the results of the earlier analysis

may be approximately applied, that for example if $K_1/K_2 = 2$,

$$\frac{a^2}{4\nu t} \doteqdot 20.$$

Suppose that the ratio of strengths between two neighbouring vortices is as great as this, then $\frac{a^2}{4\nu t} \doteqdot 20$. The distance between two vortices may reasonably be taken to be of the order of thickness of the boundary layer so that $\frac{a}{l} \doteqdot 10^{-2}$, where l is the length of the body; then the time taken for coalescence is at least

$$\frac{a^2}{80\nu} = \frac{10^{-4} l^2}{80\nu}.$$

The critical state will occur when this quantity is of the same order of magnitude as $2S/V$,

$$\frac{10^{-4} l^2}{80\nu} = \frac{2S}{V} = \frac{l}{V},$$

assuming that the eddies leave the surface approximately half way to the tail.

Thus

$$\frac{Vl}{\nu} = 8.10^5.$$

This of course is possibly much beyond the region of periodicity, since for smaller values of Vl/ν the time occupied by the vortices in transit will be sufficient to allow only a few to coalesce, and it is to be noticed that since we have assumed a/l to be of the order of 10^{-2} the number of elementary vortices must be of the order of a hundred.

In a recent paper * Relf and Simmons have determined the variation of periodicity in eddy formation for circular wires over a wide range of values of Vl/ν. For values of Vl/ν up to 10^5 they find distinct periodicities, but beyond that range nothing definite could be observed. The present investigation is in conformity with these results, and suggests that the turbulent motion in the rear of the body associated at lower speeds with the discharge of eddies at the edge must have given way to two sharp bands or trails of smaller eddies separating the wake from the rest of the fluid.

* Phil. Mag. vol. xlix. (Feb. 1925).

LXXVIII. *Orifice Flow as affected by Viscosity and Capillarity.* By H. W. Swift, *M.A., B.Sc.**

Introduction.

Apart from its theoretical interest the orifice has important possibilities as a sensitive method for the continuous measurement of liquid flow, the volumetric rate of flow being given by

$$Q = C_d a \sqrt{2gH},$$

where H is the head effective at the orifice, a its area, and C_d the Coefficient of Discharge. The practical problem of the orifice lies in the evaluation of this coefficient for the conditions under which it is required to operate, and the present limited employment of the orifice method is due largely to uncertainty in the prediction of this coefficient and its systematic variations. In the present treatment an attempt is made to examine some of the more important conditions affecting the flow and their effects, and to rationalise the consequent variations in the coefficient of discharge. Particulars of the methods adopted in the experimental work will be found at the end of the paper.

The following symbols are used :—

H, pressure head across orifice;
p, pressure differences across orifice ;
v, velocity (at vena contracta unless otherwise stated);
w, weight of liquid per unit volume ;
ρ, density ;
μ, viscosity ;
ν, kinematic viscosity ;
σ, surface tension ;
f, acceleration.

Other symbols are defined as they occur.

The English gravitational system of units is adopted as convenient in engineering applications, but the usual convention is observed in the definition of kinematic viscosity, $\nu = \frac{\mu}{\rho} = \frac{\mu g}{w}$.

In converting to absolute units the only substitutions necessary are ρg for w and $\frac{p}{\rho g}$ for H.

* Communicated by the Author.

Summary of Results.

The main points which it is hoped to demonstrate are as follows.

In a complete account of free orifice discharge as applied to liquids regard must be paid to each of the dimensionless products

$$\xi = \frac{H}{l}, \quad \eta = \frac{\nu}{lv}, \quad \zeta = \frac{\sigma}{wl^2}.$$

Of these, the product η is generally by far the most important in its effects. An increase in the value of η, exemplified by increasing viscosity, will necessarily diminish the coefficient of velocity C_v, but will always tend to increase the coefficient of contraction C_c towards the limit of unity. The coefficient of discharge $C_d = C_v C_c$ will for a re-entrant mouthpiece necessarily increase with increasing viscosity so long as the conditions of free discharge are maintained ; for sharp-edged plane orifices the coefficient increases with η up to a certain point and subsequently falls away towards zero ; for a rounded orifice it will fall continuously with increasing viscosity. In each case the relation between C_d and η over the useful hydraulic range is practically linear :

$$C_d = C_0 \pm \kappa\eta.$$

The effect of ξ is generally small and for either hydraulic or very viscous flow may usually be neglected. Over the intermediate range it may become important, but it appears rather to modify the relation between C_d and η than to alter its main characteristics.

Capillarity, represented by the product ζ, always tends to reduce the velocity and at the same time to increase the area of the contracted vein. In the re-entrant mouthpiece it has no resultant effect on the coefficient of discharge, but in other circular orifices there always follows a diminution in discharge, which will usually be small and for hydraulic purposes may be allowed for satisfactorily by a correction of the head :

$$h = H - \frac{4\sigma}{wd_c}(C_0 - \tfrac{1}{2}).$$

The Criteria governing Discharge.

The principle of dimensional homogeneity when applied to orifice flow affords qualitative results which simplify to some extent the laws controlling discharge.

The general theorem due to Riabouchinsky * shows that if $Q_1 Q_2 \ldots Q_n$ are physical quantities which completely determine a phenomenon, in the specification of which there are k fundamental units, and if the quantities $Q_1 \ldots Q_k$ are independent, different in kind and not interderivable, the general relation

$$F(Q_1, Q_2, \ldots Q_n) = 0$$

can be simplified into the form :

$$f(\Pi_1, \ldots \Pi_{n-k}) = 0,$$

where Π_x is a dimension product of the form :

$$Q_x(Q_1^{\alpha} Q_2^{\beta} \ldots Q_k^{\kappa}).$$

It follows that the quantity Q_x may be written :

$$Q_x = P_x \cdot \phi(\Pi_1, \ldots \Pi_{n-k-1}),$$

where P_x is a product of the quantities $Q_1, \ldots Q_k$ having the same dimensions as Q_x.

In the application of this theorem, the variables must be chosen with due regard to the physical circumstances, and the only checks we have upon their completeness and correctness are furnished by experimental confirmation of results and by the compatibility of the dimensional relations. In many cases of practical interest it is unlikely that all the relevant quantities will have similar importance, and it is reasonable to assume that if for simplicity certain quantities are omitted, then provided these are judiciously chosen the results will be nearly correct, in the same way that similar well-chosen omissions will not prejudice unduly the results of quantitative analysis. Exclusion from the original list, however, of dimensional quantities (such as the gravitational constant) which are constant under any or all the conditions of experiment leads to logical absurdity, for it would follow that the variation of any single dimensional quantity could have no effect on the phenomenon. A safer procedure is to include all relevant quantities, constant or variable, to proceed with the dimensional considerations and have regard to the constancy of these quantities in the final result.

In the case of free orifice flow (which for present purposes includes flow over notches or weirs of any form) the quantities

* 'L'Aerophile,' 1907, p. 407. *Cf.* also *Bull. Inst. Aerod. Koutchino*, i.–iv.

which appear to affect discharge and completely to determine it are :

$$l_1, l_2, \ldots l_n, \quad \mathrm{H}, v, w, \mu, \sigma, f,$$

and any quantity Q related to the flow as a whole may be written :

$$\mathrm{Q} \equiv \mathrm{F}(l_1, l_2, \ldots l_n, \mathrm{H}, v, w, \mu, \sigma, f).$$

A quantity relating to a particular point is likewise defined by

$$\mathrm{Q}' \equiv \mathrm{F}'(x, y, z, l_1, l_2, \ldots l_n, \mathrm{H}, v, w, \mu, \sigma, f).$$

Here H, v, w, μ, σ, f have the significance shown on p. 852, and such quantities as v, f which are variable under the conditions of a single conformation have representative values. The quantities $l_1 \ldots l_n$ are measures of length which determine the geometrical conformation of the efflux arrangements. In comparing for example sharp rectangular orifices of varying proportions, provided the other dimensions be varied in the corresponding ratios, two quantities $l_1 l_2$ will be necessary and sufficient. Moreover, if the geometrical conformation remains similar, this similarity extending to conditions of approach, roughness, and depth of liquid above orifice, then it may be specified by a single parameter l.

In the statement of variables no assumption is made or implied that any one of the quantities introduced is independent of the other conditions. It is merely assumed that its value is always determinable from some representative value by means of the other variables mentioned. For example, the possibility that the viscosity may vary with the rate of distortion will not vitiate the argument provided that its value at a definite rate of shear be taken as standard. For we have

$$\mathrm{Q} = \mathrm{F}(l, \mathrm{H}, v, w, \mu, \sigma, f),$$

and supposing

$$\mu = \phi(l, \mathrm{H}, v, w, \mu, f)$$

we can by eliminating μ obtain

$$\mathrm{Q} = \mathrm{F}'(l, \mathrm{H}, v, w, \mu_0, \sigma, f).$$

The acceleration f at corresponding points we may regard as dependent upon the other variables and upon the gravitational constant g :

$$f = \phi_1(\mathrm{H}, l, v, w, \mu, \sigma, g).$$

In the same way the velocity may be regarded as dependent

upon its value at an agreed point of reference under standard conditions and upon the other variables. Choosing the point of reference at the vena contracta and the standard conditions those given by a 'perfect' liquid, we have:

$$v = \phi_2(\mathrm{H}, l, v_0, w, \mu, \sigma, f).$$

Hence any quantity Q relating to the flow as a whole, by virtue of these two relations, may be written without loss of generality:

$$\mathrm{Q} = \mathrm{F}_0(\mathrm{H}, l, v_0, w, \mu, \sigma, g).$$

But a definite physical relation exists between certain of these quantities:

$$v_0^2 = 2g\mathrm{H}.$$

Hence we may regard either of these quantities as redundant in the general expression, and may write at discretion:

$$\mathrm{Q} = \mathrm{F}_1(\mathrm{H}, l, \mu, w, \sigma, g) = \mathrm{F}_2(v, l, \mu, w, \sigma, g).$$

For the present it will be convenient to consider the former expression, F_1.

By virtue of dimensional equations the six variables in this expression will furnish the three dimensionless products:

$$\xi = \frac{\mathrm{H}}{l}, \quad \eta = \frac{\nu}{l\sqrt{g\mathrm{H}}}, \quad \zeta = \frac{\sigma}{wl^2}.$$

If the head be replaced by the pressure difference and absolute units are adopted, these criteria take the form:

$$\xi = \frac{p}{l\rho f}, \quad \eta = \frac{\nu}{l}\sqrt{\frac{\rho}{p}} \quad \zeta = \frac{\sigma}{\rho g l^2}.$$

Now if Q is the value at the point x, y, z of any quantity scalar or vector related to the motion, and Q_0 is the value of this quantity at the corresponding point x_0, y_0, z_0 under any conditions of flow which may be chosen as a basis of comparison, then we may write

$$\mathrm{Q} = \mathrm{P}f\left(\frac{x}{l}, \frac{y}{l}, \frac{z}{l}, \xi, \eta, \zeta\right),$$

$$\mathrm{Q}_0 = \mathrm{P}_0 f\left(\frac{x_0}{l_0}, \frac{y_0}{l_0}, \frac{z_0}{l_0}, \xi_0, \eta_0, \zeta_0\right),$$

where P is a product of the relevant quantities of the same dimensions as Q. Hence the relations:

$$\xi = \xi_0, \quad \eta = \eta_0, \quad \zeta = \zeta_0$$

provide the necessary and sufficient conditions that $\frac{Q}{Q_0}=\frac{P}{P_0}$ and in particular that any dimensionless quantity $\Pi=\Pi_0$. This result embraces of course the Principle of Dynamical Similarity.

The coefficient of discharge from an orifice is a dimensionless quantity, and it may therefore be expressed in the form :

$$C_d = f\left(\frac{H}{l}, \frac{\nu}{l\sqrt{gH}}, \frac{\sigma}{wl^2}\right) = f(\xi\,\eta\,\zeta).$$

The Ideal Liquid.

In the case of the ideal liquid μ and σ vanish, and the expression for the coefficient of discharge reduces to

$$C_d = f_1(\xi)$$

when a single parameter defines the configuration, and to

$$C_d = f_1\left(\frac{l_2}{l_1}, \frac{l_3}{l_1}, \ldots \frac{l_n}{l_1}, \xi\right)$$

in the more general case. The coefficient of discharge for a 'perfect liquid' therefore is independent either of its density or of the gravitational system. There is no dimensional reason, however, why C_d should be constant at all heads for a given geometrical configuration. The question must be decided by other physical considerations or by experiment.

The General Case.

In the general case we have three arguments in the expression for C_d. For strict similarity in a single gravitational system between two motions M_1 M_2 which are controlled by the quantities under present consideration, the following relations must therefore hold :

$$\xi_1 = \xi_2, \quad \eta_1 = \eta_2, \quad \zeta_1 = \zeta_2.$$

These equations require

$$\left(\frac{\nu_1}{\nu_2}\right)^4 = \left(\frac{\sigma_1}{\sigma_2}\cdot\frac{w_2}{w_1}\right)^3.$$

This relation between their physical properties is of course in general not satisfied by two different liquids, and it follows that we cannot expect strict similarity between two flows of different liquids, and since the conditions also require

$$\frac{\nu_1}{\nu_2} = \left(\frac{H_1}{H_2}\right)^{3/2} = \left(\frac{l_1}{l_2}\right)^{3/2}$$

it follows that no two different motions with a single liquid can be similar.

The general result, containing as it does three separate arguments, is complicated and unlikely to yield results of practical value as it stands. It indicates the direction in which the laws of flow may most profitably be explored, but it affords no evidence of their form. In order to obtain information on this point, as a basis for the interpretation of experimental results, it will be well to consider separately the effects of the three criteria ξ, η, ζ. In most practical applications of the orifice the effect of these factors is known to be comparatively small, and it will be of some value if even an approximation to the law of discharge can be found to cover this range with sufficient accuracy and indicate under what conditions any or all of the effects may justifiably be neglected.

Effect of Surface Tension.

In all cases of submerged discharge, and in free discharge from large orifices, the effect of capillarity is not likely to be important. It seems reasonable in such cases to neglect it in the general expression, which then reduces to

$$C_d = f(\xi, \eta).$$

The effect of surface tension, however, on the free discharge from small orifices is not negligible. It will in the case of a circular orifice give rise to a pressure difference across the surface of the jet which can be readily calculated at the vena contracta :

$$2\sigma = \Delta p \,.\, d_c = h_s w d_c.$$

The pressure inside the jet at the vena contracta will therefore be increased and the effective head causing discharge reduced by the amount :

$$h_s = \frac{2\sigma}{wd}.$$

In the case of water this correction represents $\frac{\cdot 022}{d_c}$ inches of head, where d_c is measured in inches at the vena contracta, and may therefore be appreciable.

For a perfectly rounded orifice a simple correction of the head will be sufficient, $h = H - h_s$; but where contraction takes place the effect is not so simple, since, although the pressure at the vena contract is increased, there is no similar appreciation over the annular section $(a - a_c)$ across which the jet contracts. The pressure distribution in the neighbourhood of the orifice will therefore no doubt be modified

to some range of values intermediate between those for non-capillary discharge at heads of H and $H-h_s=h$. In this way there will be induced an effect upon the contraction of the jet. The force causing discharge of a perfect liquid may be written :

$$P = wH(a+\Delta a), \text{ where } 0 \leq \Delta a \leq a.$$

Admitting capillarity but not viscosity, the discharge is determined by

$$\frac{w}{g} C_c a v^2 = wHa + wH'\Delta a - wh_s C_c a,$$

where

v = velocity at vena contracta $= \sqrt{2gh}$.

$h_s = \frac{2\sigma}{wd_c}$.

$wH'\Delta a$ = resultant deficit of pressure on wall in the plane of the orifice.

By hypothesis,

$$wH \,.\, \Delta a \geq wH' \,.\, \Delta a \geq wh \,.\, \Delta a.$$

Hence it follows :

$$C_c = \frac{Ha + H'\Delta a}{2ha + h_s a},$$

where

$$C_0 = \frac{a+\Delta a}{2a}.$$

Writing $H' = H - \lambda h_s$, where $0 \leq \lambda \leq 1$,

we obtain by substitution :

$$C_c = C_0 + \frac{h_s}{2H - h_s}(C_0 - \lambda\sqrt{2C_0 - 1}),$$

and necessarily :

$C_c > C_0$ provided C lies between 0·5 and 1·0.

Now the coefficient of velocity $C_v = 1 - \frac{h_s}{2H}$ nearly.

Hence

$$C_d = C_c C_v = C_0 - \frac{\lambda h_s}{H}(C_0 - \tfrac{1}{2}).$$

It follows that surface tension through its effect upon the contracted vein itself will tend to increase the size of the jet, but when $C_0 > \frac{1}{2}$ will necessarily diminish the discharge. This diminution will not be greater than $-\delta C_d = \frac{h_s}{H}(C_0 - \frac{1}{2})$, which for water, taking C_0 as 0·65, gives a maximum value of about $\frac{\cdot 004}{d_{\text{in}}}$ when d and H are measured in inches.

Some estimate of the probable value of the effect is obtained by regarding the distribution of pressure in the approach chamber and in the neighbourhood of the orifice as that corresponding to the mean discharging head across the orifice:

$$H' = \frac{H(a - a_c) + (H - h_s)a_c}{a} = H - C_0 h_s.$$

In this case $\lambda = C_0$ and the *fractional* reduction in discharge will be $\frac{h_s}{H}(C_0 - \frac{1}{2})$ nearly, which for hydraulic conditions may be taken as $\frac{\cdot 004}{d\sqrt{H}}$. This effect is usually small and may be satisfactorily eliminated by applying a constant correction of $-\frac{\cdot 008}{d}$ inches to the head. It is this correction which has been applied when necessary in the present experimental work.

There will of course be other effects of capillary forces, but they are not easy to separate and evaluate and are likely to be still smaller than that already considered. The general manner in which they will modify the discharge is clear. For a given liquid the effective force will by its nature depend upon the geometry of the system and will be almost independent of the head. The effect of this force on the discharge will therefore vary inversely with the head, and any modification of the coefficient will be of the form $C = C_0 \pm \frac{q}{H}$ where q is of linear dimensions, and to be identified with the particular orifice and liquid.

In the case of the Borda re-entrant mouthpiece it is interesting to observe that surface tension can itself have no effect upon the coefficient of discharge. For the force causing discharge will be

$$wa(H - C_c h_s) = \frac{w}{g} C_c a v^2,$$

and since $C_v = 1 - \frac{h_s}{2H}$ it follows that $C_d = C_c C_v = \frac{1}{2}$, irrespective of capillary effects.

Effect of Variations in Head.

Correct treatment of the effect of variations in head requires some care. The general expression for the coefficient of discharge has reference only to flows in which

the artificial geometrical configuration may be defined by a single parameter l. In many discharge arrangements commonly adopted the surface of the liquid in the approach chamber is free to atmospheric pressure. The depth of the liquid above the orifice will therefore vary with the head, so that simple variations in head will necessarily vitiate the conditions of strict similarity. It is found from experiments with a single orifice in a horizontal plane, discharging under constant head but with varying depth of water, that the effect of this change is small but measurable and that discharge increases with the depth. It would therefore appear that although with varying depths the geometrical configuration may not be accurately represented by the single parameter l, results obtained by ignoring such variations in depth are likely to be seriously in error only in exceptional cases. Moreover, dimensional considerations allow us to write in this case:

$$C_d = f\left(\frac{H}{D}, \frac{H}{l}, \eta, \zeta\right),$$

where l refers to the orifice and to the plan dimensions of the approach chamber. In all experiments with surface free to atmosphere, $H = D$. Hence, in spite of the absence of 'artificial similarity,' we may write for such experiments:

$$C_d = f_1(\xi, \eta, \zeta)$$

which function will be theoretically distinct from, but practically similar to, the corresponding expression

$$C_d = f_2(\xi, \eta, \zeta)$$

which defines the discharge when the geometry of approach is the same at all heads and the quantity l is fully representative. The experiments to which reference will shortly be made were nearly all of the free surface type, so that it is the function f_1 which is strictly speaking under discussion.

Variations in the value of the head H will affect both the remaining criteria of flow, ξ and η. Its effect through the latter will be considered in connexion with viscosity; at present attention will be confined to the former, $\xi = \frac{H}{l}$. The necessity for this criterion has not always been admitted. In the study of the flow of liquids in pipes it is found unnecessary in general to introduce the gravitational constant g, and in certain published work on liquid efflux under gravity it has been omitted. The simplification achieved is considerable, for if surface tension can be allowed for or

neglected the variables may then be reduced to l, v, μ, w, which afford the single dimensionless product $\frac{\nu}{lv}$, and ξ disappears. The coefficient of discharge is then necessarily of the form $C_d = f\left(\frac{\nu}{lv}\right)$ and a single series of determinations, made for example by varying the head with other conditions fixed, would furnish this function and enable the value of the coefficient for any similar configuration under any conditions to be obtained from a single curve. The question is therefore of some importance.

For a liquid without capillarity or viscosity the coefficient is necessarily of the form $C = f\left(\frac{H}{l}\right)$. Under the hydrodynamical conditions of efflux from the slit orifice and re-entrant mouthpiece it is known that the coefficient has a definite value irrespective of the velocity of discharge and therefore of the head; and there is no physical reason for anticipating departure from this constancy in orifices of other forms. In nature however, apart from 'imperfections' in the liquid, these hydrodynamical assumptions are not strictly tenable.

(*a*) When liquid discharges from an orifice in a vertical plane the head will not be the same for each individual filament, and when measured in the conventional manner from the centroid of the orifice will not accurately represent the mean effective head. Corrections calculated on a rational basis have been proposed * for the rectangular and circular orifice, and these to the first significant order are of the form:

$$C_d = C_0 - \gamma\left(\frac{r}{H}\right)^2,$$

where $\left\{\begin{array}{l} r \text{ is the semi-vertical dimension of the orifice,} \\ H \text{ is the head measured from the centroid of the orifice.} \end{array}\right.$

The correction is found in each case to be negligible when the ratio $\frac{r}{H}$ is less than $\frac{1}{4}$; but for large orifices it increases rapidly in importance at low heads, and it is known that under these circumstances the coefficient may actually fall with the head.

(*b*) When an orifice is mounted horizontally the head is conveniently measured from the plane of the orifice, but here the jet is falling during contraction and under ordinary heads

* C. Monteil, *Ann. d. Phys. et Chim.* xxvii. (3) No. 35, p. 139 (1907).

values of the coefficient so deduced are not likely to correspond with those obtained when the orifice is vertical. Experiment shows that this is the case, and also indicates that for sharp-edged orifices up to 1·0 inch in diameter a satisfactory working agreement is obtained if the head is measured from a point about $\frac{d}{6}$ below the plane of the horizontal orifice.

In the case of the ideal liquid, however, it seems likely that the effect of variations in ξ will be small and usually negligible. It does not necessarily follow that this is the case when viscosity and capillarity are admitted. For guidance at this point we shall have recourse to experiment.

Experiments have been made by G. F. Davidson * in the hope of establishing η as the sole criterion with submerged rounded orifices. He worked with an oil at different temperatures and found that the value of this argument did not fix the coefficient of discharge. The physical peculiarities of the oil used, however, give rise to some uncertainty as to the correct values of the viscosity to be applied, and the evidence cannot be regarded as conclusive. W. N. Bond †, working with mixtures of glycerine and water through a sharp-edged drowned orifice, and assuming η as the criterion of flow, obtained results differing considerably from Davidson's and showing variations in themselves up to 3 per cent. from the mean curve. On the other hand, determinations by J. L. Hodgson ‡ of the discharge of air and water through a single orifice in a pipe-line indicate that for what may be termed limpid flow the quantity η, at least up to the limits of experimental accuracy, is a sufficient criterion of the motion. The balance of existing evidence tends therefore to show that the error involved by the omission of the gravitational constant is not large, but it cannot be regarded as established that such an error does not exist, and it was thought well not to dismiss it without confirmation.

Three series of experiments were made bearing upon this point, under the conditions of limpid, fairly viscous and very viscous flow. In each case the discharge was free.

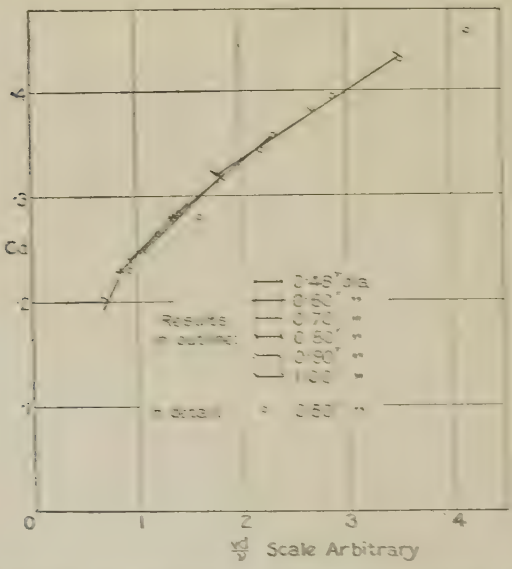
The first series of tests was carried out with an oil of similar properties to that employed by Davidson, having a high viscosity of about $50 \cdot 10^{-3}$ ft.lb.sec. units at 15° C. Variations in the value of the product $\left(\frac{\nu}{lv}\right)$ were obtained

* Proc. Roy. Soc. lxxxix. p. 91 (1913).
† Proc. Phys. Soc. xxxiii. p. 225 (1921).
‡ Proc. Inst. C. E. cciv. p. 192.

by the use of orifices of various diameters from 0·48 to 1·0 inches and by varying the head. The results which are plotted in fig. 1 indicate that for very viscous discharge the single criterion η determines the flow with sufficient accuracy, and that the inclusion of ξ is unnecessary.

In the second series water was used, and a range of values of η was obtained by varying the temperature and head with a given orifice. Two orifices 0·3 and 0·5 inch in diameter were employed and values of the coefficient

Fig. 1.



plotted against η as shown in fig. 2. The results at higher temperatures, possibly owing to the increased difficulty of avoiding turbulence and to inequalities of temperature in the orifice plate and in the liquid, were not so consistent as with cold water. When corrected for attitude and surface tension, however, they show no more serious departure from a single curve than might be due to experimental errors, and it seemed reasonable to conclude that for limpid discharge the criterion ξ could be omitted without serious error.

There was, however, in the results some indication of a systematic departure from the single curve, and a third series of tests was therefore made to cover the range of fairly viscous flow. A single orifice 0·1 inch in diameter

was employed, and determinations were made with six different liquids varying in viscosity from 2.10^{-3} to $\cdot 024.10^{-3}$ ft.lb.sec. units and under heads between 5 feet

Fig. 2.

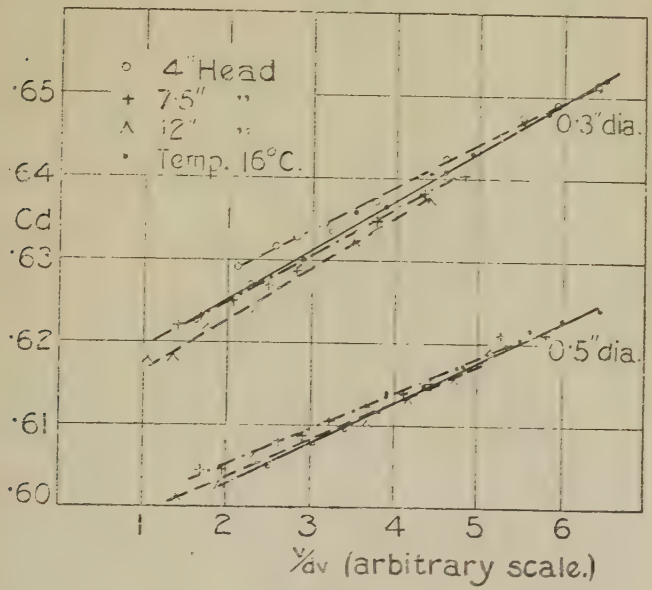
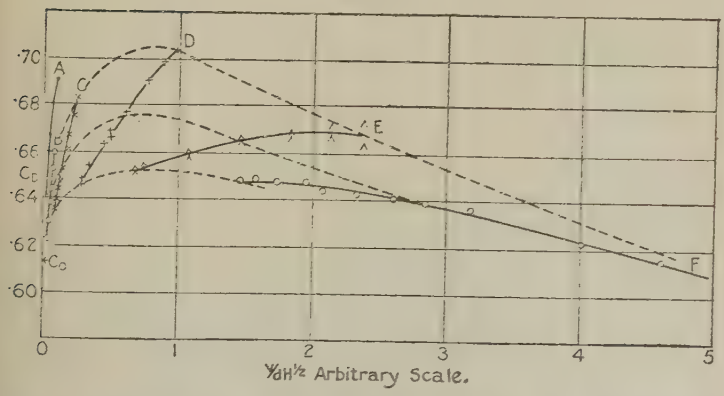


Fig. 3.



and 6 inches. The results are plotted in fig. 3. They show a distinct systematic divergence between the curves obtained with the various liquids, and indicate clearly that over this particular range the discharge is not controlled by the single

criterion η. The results have been left uncorrected for capillarity for the sake of clearness. Correction would further accentuate their divergence.

The physical explanation of these results is not altogether clear, but the fact that all the curves may reasonably be considered to converge upon a single point in the limit when $\eta=0$ indicates that the divergence is not to be attributed to any variation in the value of the coefficient for a perfect liquid, and the fact that agreement becomes good again for very viscous flow, where in point of fact there is practically no contraction of the jet, suggests that the effect operates by modifying the contraction in viscous flow, and becomes small with either the viscosity or the contraction. Whatever the nature of the effect, the results furnish definite evidence that not only is it theoretically unwise to omit the gravitational constant in dimensional analysis without examination, but that such omission may lead under certain circumstances to serious errors in application. These results refer only to the case of the free discharge of a liquid into the air, and without further experimental work it would be unsafe to apply them to the conditions of submerged discharge.

So far as the conditions of hydraulic discharge are concerned, however, the divergence is not serious, and a single curve connecting the coefficient with the values of η could be made to cover with good accuracy the range of viscosities over which water is likely to vary in practice. Moreover, the variations indicated in fig. 2 refer to smaller orifices than are normally employed, and agreement will improve as the size of the orifice is increased.

Effect of Viscosity.

From considerations already advanced it will be clear that the criterion $\eta=\frac{\nu}{lv}$ * has by far the greatest influence upon the value of the coefficient of discharge, and it is proposed to seek now an approximate relation between the two which shall hold when ξ and ζ are constant or inoperative. The treatment is frankly of a rational nature. The assumptions made are in the main based on known empirical facts, while those specially introduced do not

* Writing $C_d=f\left(\frac{H}{l}, \frac{\nu}{lv}, \frac{\sigma}{wl^2}\right)$.

appear unreasonable and will rely for support upon experimental confirmation of the results.

For simplicity we shall consider in the first instance flows in which H, l, $\frac{\sigma}{w}$ have constant values, so that the kinematic viscosity will be the only independent variable under consideration and the criterion of motion will be $\frac{\nu}{lv}$.

By experimental tests it is found that the total loss of head H_x during approach and discharge is for both water and very viscous liquids proportional to the velocity at the vena contracta. If, therefore, in the criterion $\frac{\nu}{lv}$ the velocity v be defined by its value V at discharge, we may write

$$H_x = \frac{V^2}{2g} \cdot a\eta,$$

where a is a positive number dependent only on the form of the outflow arrangements and the particular values of ξ and ζ. From this it follows at once that the velocity of discharge is given by

$$V^2(1+a\eta) = V_0^2,$$

so that the coefficient of velocity follows the law:

$$C_v = (1+a\eta)^{-\frac{1}{2}},$$

and will of course decrease continuously as the viscosity becomes greater.

Consider now a surface drawn normal to the stream-lines and containing the bounding circle of the orifice. The energy equation at any point on this surface may be written in a common notation:

$$\frac{p}{w} + z = H - \frac{v^2}{2g} - h_x,$$

where v is the velocity at the particular point and h_x the loss of head incurred by the liquid in its movement up to that point. It seems reasonable to assume for descriptive purposes that this loss may be written:

$$h_x = \frac{v^2}{2g} \cdot b\eta,$$

where b is in nature similar to a. Now the curvature of the stream-lines is determined by the pressure gradient

across them. At the surface under consideration it follows that

$$\frac{v^2}{g\rho} = \frac{\partial}{\partial n}\left(\frac{p}{w}+z\right) = -(1+b\eta)\frac{\partial}{\partial n}\left(\frac{v^2}{2g}\right).$$

But the velocity v will be less than the corresponding velocity v_0 under non-viscous conditions, and the two are likely to be related in a manner similar to V and V_0:

$$\frac{v^2}{2g} = \frac{v_0^2}{2g}\cdot\frac{1}{1+c\eta}.$$

Hence the curvature will be given by:

$$\frac{v_0^2}{g\rho} = -(1+b\eta)\frac{\partial}{\partial n}\left(\frac{v_0^2}{2g}\right),$$

where

$$\frac{v_0^2}{g\rho_0} = \frac{\partial}{\partial n}\left(\frac{v_0^2}{2g}\right),$$

and we have:

$$\frac{\rho}{\rho_0} = \frac{1}{1+b\eta}.$$

The coefficient of contraction C_c is related to the curvature of the stream-lines at discharge by some expression of the form:

$$C_c = \left(1-e\cdot\frac{\rho}{l}\right)^2.$$

By substitution it follows that

$$1-C_c = \frac{1-C_0}{1+b\eta} - b\eta\left(\frac{1-C_0}{1+b\eta}\right)^2,$$

where C_0 is the theoretical coefficient for the orifice, and since b is necessarily positive this is a continuously decreasing function so long as $C_0 < 1$.

The value of the coefficient of discharge will be obtained by multiplying C_c by C_v. When the value of η is small we can obtain simple approximations to the coefficients:

$$C_c = C_0 + b\eta(1-C_0)(2-C_0), \quad C_v = 1-\frac{a\eta}{2}.$$

Whence it follows:

$$C_d = C_0 + b\eta(1-C_0)(2-C_0) - \frac{C_0 a\eta}{2} \text{ nearly.}$$

Whether $C_d \gtrless C_0$ under these circumstances depends upon the comparative values of a and b and is not easily determinable.

In the case of a Borda mouthpiece, for which C_0 is 0·5, the question is settled definitely by the principle of momentum. The force causing discharge will be the unbalanced pressure P over the area of the orifice together with a frictional force F acting along the surface of contact of the re-entrant tube and the liquid. The frictional force will act against the direction of motion of the liquid along this surface, and therefore will actually assist the discharging force P. The principle of momentum then requires:

$$wHA + F = \frac{w}{g} A V_0^2 C_d C_v,$$

where

$$wHA = \frac{w}{g} A V_0^2 C_0,$$

and it necessarily follows that

$$C_d C_v > C_0$$

and *a fortiori*

$$C_d > C_0.$$

Discharge will therefore increase with increasing viscosity under all conditions so long as the mouthpiece will run free.

In the case of a perfectly rounded orifice the coefficient of contraction is always unity so that the coefficient of discharge will fall continuously. In intermediate cases the instances cited and expressions obtained show that the probability of an initial increase in C_d with η is much greater with low than with high values of the theoretical coefficient C_0.

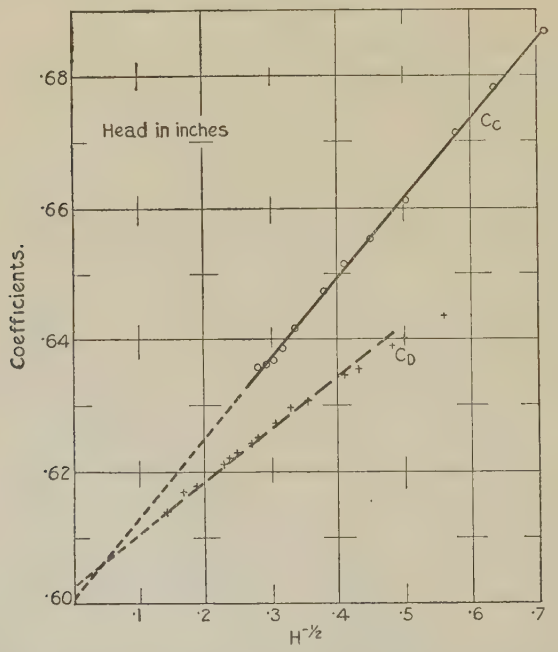
The considerations just advanced have been based on variations in viscosity alone, but the dimensional analysis shows that the results must necessarily hold in the same way for changes in the value of η from any cause which does not affect the values of ξ or ζ. For flows in which this condition holds, or in which these factors are inoperative, we are led to the following conclusions :—

1. An increase in the value of η will always tend to increase the coefficient of contraction from its value C_0 for a perfect liquid towards a maximum of unity. In the case of very viscous liquids we may reasonably expect therefore that contraction will be almost completely suppressed and the discharging jet will have the same dimensions as the orifice. On the other hand, for limpid liquids the expression suggests that over a certain range the coefficient will follow approximately a linear relation :

$$C_c = C_0 + \kappa\eta.$$

Determinations have been made to verify these results. With a very viscous oil discharging from sharp orifices 0·3 and 0·5 inch in diameter no contraction could be detected at heads below about 18 inches. Accurate measurements, however, were difficult to make on account of the adhesive nature of the oil. At greater heads the contraction could be measured, and was found in the case of the 0·3 inch orifice to vary in a manner consistent with the general relation expected to a value of 0·81 under a head of 46 feet.

Fig. 4.

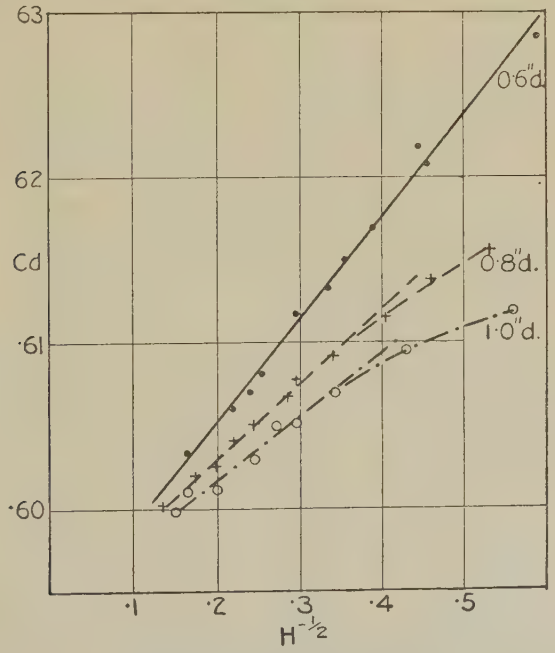


More accurate measurements were made upon the discharging jet of water from an orifice 0·48 inch in diameter, under heads from 2 to 15 inches, The results are plotted in fig. 4, and show the linear relation quite clearly. The linearity is probably helped in this case by the fact that the effect of surface tension on the contraction tends to balance the drooping of the line at low heads, but it is none the less noteworthy. Under the conditions of hydraulic discharge the values of η will normally be still lower, and the linear relation may therefore be safely applied.

2. The coefficient of discharge is dependent upon the value of the coefficient of contraction which continuously

increases and the coefficient of velocity which continuously decreases as η becomes greater. For a sharp orifice the form of the expressions obtained suggests that increments in C_c may at first be greater than corresponding decrements in C_v and the coefficient of discharge will rise to a maximum, beyond which changes in C_v will predominate and it will fall towards zero as a limit. For a rounded orifice changes in C_v will predominate throughout and the value of C_d will fall continuously with increasing η. The earlier part of the curve we may expect in either case to be approximately linear.

Fig. 5.

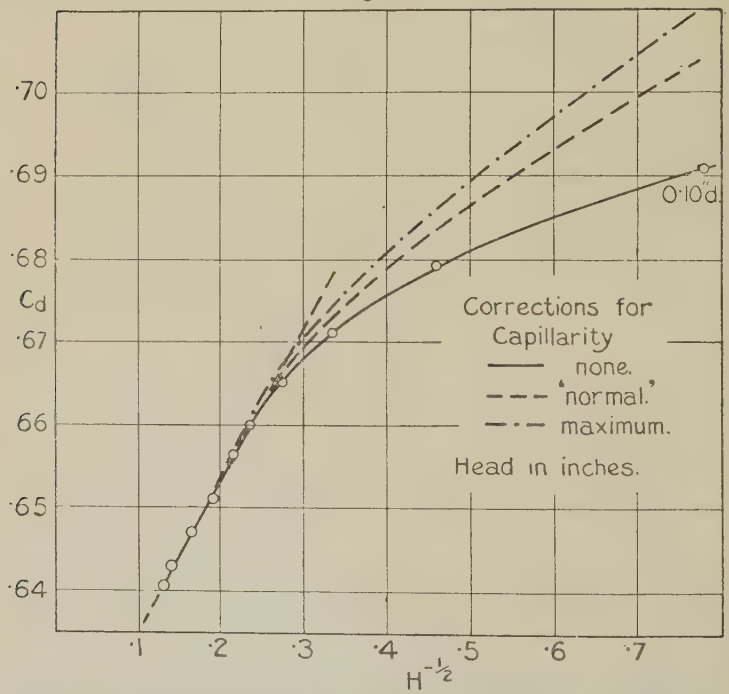


A number of experiments were made to verify these conclusions. In these it was assumed that for hydraulic flow and very viscous flow the effect of ξ was unimportant, so that changes in head were admissible in varying the value of η.

(a) Determinations of the coefficient of discharge for water from sharp orifices of 0·48 to 1·0 inch in diameter under heads varying from 4 inches to 4 feet showed that the variations were generally in good agreement with the relation expected, corrections having been applied where necessary for capillarity. The collinearity of these

results, which are shown in figs. 4 and 5, is at low heads more satisfactory with small than with larger orifices, and with the 1·0 inch orifice the coefficient shows at heads below about 5 inches a distinct tendency to fall. The head in this case is no longer great compared with the diameter of the orifice, and the graduation in head across the section of the orifice begins no doubt to be effective. At the greater heads common in practice collinearity is very marked even without correction for surface tension and shows to better advantage on an open

Fig. 6.



scale. It is noteworthy that these experiments afford no evidence in support of a 'critical head' and a subsequently constant coefficient, and the conditions of viscous flow are easily maintained up to a value of Reynolds' criterion $\frac{vd}{\nu} = 126 \cdot 10^3$ as compared with the value of 2000 which roughly limits the viscous condition in uniform pipe flow.

(*b*) The extent to which the linear relation holds was investigated by determinations of the discharge of water through a smaller orifice about 0·1 inch in diameter. The

results (fig. 6) show that, assuming the correction for surface tension is adequate, divergence becomes evident with this orifice at a head of about 10 inches, and since the value of the corresponding head will vary inversely with the square of the diameter it is clearly outside the normal range of hydraulic application. The deviation of the curve from a straight line is in this case to be attributed to the fact that as the Coefficient of Contraction increases towards the limit of unity its rate of increase becomes slower and the Coefficient of Discharge tends to approach its maximum value and subsequent fall.

(*c*) Measurements of the coefficient of discharge under very viscous conditions have already been referred to (fig. 1), and these agree reasonably well with the form expected.

(*d*) The manner in which the C_d, η curve passes over its maximum for a fixed value of the criterion ξ is shown by the dotted lines in fig. 3.

Details of Experimental Methods.

a. *Experiments with Water.*

Coefficient of Contraction.

The orifice used was 0·4805 inch in diameter, with edge as sharp as possible bevelled at 45°. The orifice plate was of brass, 8 inches diameter, $\frac{3}{8}$ inch thick, and polished. It was mounted in a vertical wall of a tank 24 inches square in plan, and approach was stilled by means of sheets of perforated metal.

Readings of head were made by means of a hook-gauge pointed upwards and a kathetometer. It was found that the hook-gauge was readily sensitive to ·001 inch in still water. The dimensions of the escaping jet were measured by four micrometer depth gauges mounted in a square frame which was adjustable in a vertical plane about an inch from the sharp edge of the orifice. The points of these gauges were sharpened, and contact with the jet was indicated by the formation of ripples on the surface. The measurements were made at times when perfect steadiness could be maintained, and the diameters are considered correct to ·02 mm. During these determinations the jet showed no visible unsteadiness at a distance of four feet from the orifice.

Coefficient of Discharge.

The same orifice was employed in the first instance, and later others of different diameters constructed in a similar way, and screwed flush into a brass orifice plate. The orifice plate was mounted in the wall of the approach chamber and the water stilled as before. Fine adjustment of supply was arranged by means of a bye-pass valve controlling the supply pressure so that the head could be maintained constant during runs.

Heads less than 20 inches were measured by hook-gauge, higher ones by the level in a glass tube connected to the tank. In each case a vernier was used reading to ·01 inch. The jet was steady during runs except in the case of the 0·9 and 1·0 inch orifices under heads of less than 8 inches, when a slight tendency to vortex motion was just discernible occasionally. A swivel pipe below the orifice enabled the tests to be started and stopped with exactitude. The quantity discharged was measured by means of a hook-gauge and vernier mounted on a calibrated rectangular tank, and to ensure consistency the initial and final levels were approximately the same in all determinations.

b. *Experiments with Light Oils.*

The determinations of the coefficient of discharge for light oils plotted in fig. 3 were obtained by timing the fall of level in a calibrated vertical pipe 6 inches in diameter. A control experiment made with water gave results in good agreement with those obtained at constant head. As affording comparative results this method is found simple, expeditious, and reliable, and perfect steadiness is easily maintained. For absolute determinations the more usual method at constant head is preferred. The oils used were of paraffin base and varied from a machine oil to 'paraffin oil' through mixtures of various proportions.

c. *Experiments with Crude Oil.*

The oil was a Mexican crude oil of sp. gr. 0·957 at 15° C., and of a similar nature to that used by Davidson. Its viscosity is high (about $50 \cdot 10^{-3}$ ft.lb.sec. units at 15° C.) and varies very considerably with the temperature. Moreover, the viscosity is also dependent on the recent history and mechanical treatment of the oil. The normal (minimum) viscosity is only attained after considerable pumping or working, and there is evidence of plasticity until this state is reached. The oil is rich in asphaltic constituents.

Physical homogeneity in the oil during a series of determinations is not easy to secure on account of its high viscosity and low conductivity, which are apt to keep local the quite sensible temperature rise due to continued circulation.

Low Heads.

The same orifices were used as with water, mounted in the same way in a vertical plane. The supply of oil was maintained by means of a ram pump and controlled by a bye-pass valve. Variations in temperature were found to exercise such an important influence that continual readings on a thermometer in the discharge jet were necessary to ensure reliable results. Temperature effects in the results were overcome by running at different temperatures and reducing to standard conditions by interpolation.

Head was measured by hook-gauge in the approach chamber. At low heads it was not possible to maintain a free jet, so that determinations were of no value. The discharge was measured as for water, but owing to the nature of the oil a similar accuracy is not to be expected, and errors up to 1 per cent. are not unlikely.

Greater Heads.

Supply was maintained by pump to a closed approach chamber, and pressure variations due to the fluctuating supply were reduced by a device which had the effect of a large air-vessel. Oil level in the approach chamber was indicated by a mercury gauge.

The head was controlled by means of the air pressure and measured on a mercury gauge connected by a short oil lead to the orifice chamber. It was found that this device followed quite closely the changes in pressure. Control of supply, measurement of discharge, and continual reading of temperature were carried out as at low heads, and in the later work the viscosity was measured during each run by means of a tube fitted to the side of the approach chamber. It was also found more expeditious to weigh the discharge rather than to measure it by hook-gauges.

The author wishes to thank the University of Leeds for facilities in carrying out this experimental work and for supplying most of the apparatus used. His personal thanks are also due to Professor Goodman for his keen interest and encouragement.

Leeds,
December 1925.

LXXIX. *Self-Reversal of the Red Hydrogen Line.* *By* R. W. Wood *.

IT is usually assumed that self-reversal of a spectrum line can occur only when a cooler or less powerfully excited layer of the luminous gas lies between the spectroscope and the source of the radiation, as for example when the yellow sodium lines of an arc or bunsen burner are reversed by the mantle of less luminous or non-luminous vapour surrounding the flame. We should not, therefore, expect reversal in the case of an end-on spectrum tube so constructed that no dead-space existed between the discharge and the exit window.

I have, however, repeatedly observed self-reversal of both components of H_α under the above specified conditions.

Experiments were in progress with a view of obtaining, if possible, the resonance radiation of excited hydrogen. A long hydrogen tube, brought to what I have called in previous papers the "black stage" (in which only the Balmer lines are seen when the tube is viewed through a prism, the region between the monochromatic images being quite black), was mounted alongside and close to a second hydrogen tube, feebly excited by comparatively low voltage, and an increased illumination looked for (end-on) when the light of the powerfully excited tube was allowed to enter it. No trace of the phenomenon was found, though Dr. McCurdy, working in our laboratory, has recently observed it in the case of helium, under similar experimental conditions, and it was most beautifully demonstrated to me recently by De Groot in the Philips laboratory at Eindhoven.

Failure to obtain any indication of resonance-radiation under any conditions,—I tried several other arrangements,—made it seem necessary to examine the conditions under which Hydrogen absorbs the light of H_α.

The phenomenon of the reversal of the hydrogen lines in the laboratory has been observed and described many times (Ladenburg and Loria, Hulburt and others). In all of these cases the conditions were not quite the same as those obtaining in the case of resonance radiation. It was necessary to see whether feebly excited, and nearly non-luminous hydrogen, could be made to absorb the two components of H_α emitted by a tube of moderate luminosity or cause their

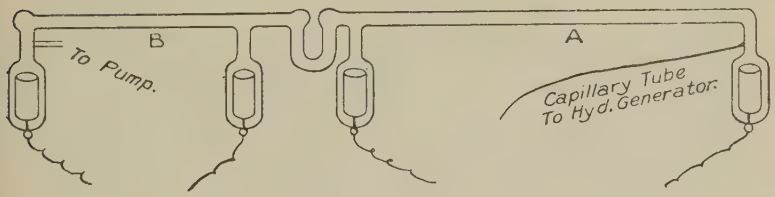
* Communicated by the Author.

reversal. I accordingly constructed a tube system as shown in fig. 1—A the emitting, B the absorbing tube.

The right-angle bends were blown out paper-thin, so that clear vision through the whole system was possible. The spectroscope and echelon grating were placed at a distance of about two metres from the end of the B tube, and its end-on image projected on the slit with a lens of 3 cms. diameter and 20 cms. focus.

This disposition is necessary if it is desired that every portion of a long tube contributes equally to the illumination of the instrument, as I have pointed out in previous papers. The tubes were excited independently by means of two high potential transformers.

Fig. 1.



With A moderately excited, so that the two components of H_α showed well separated in the echelon, it was found impossible to reduce their intensity or cause reversal by a feeble discharge in B. If, however, the current in A was augmented until the components fused together, and a current of equal intensity passed through B, H_α became a wide hazy line with two distinct darker lines in the position previously occupied by the two components. It was then found that the reversal of the two components could be effected without employing the tube B at all, by merely increasing still further the current in A. In other words, self-reversal appeared under conditions precluding the existence of an absorbing layer under physical conditions different from those of the emitting layer.

By no current adjustment could the reversed components be made really dark. They were dark only in comparison to the very bright field on which they were projected.

The reversal takes place in the following way I believe. It is well known that the intensity of the light emitted by a

gas which both emits and absorbs a radiation of given wave-length increases as the depth of the radiating layer increases, reaching a final maximum value defined by $\frac{E}{A}$, beyond which point a further increment in the thickness of the layer contributes nothing to the illumination. This is the illumination due to a layer of infinite thickness, but in practice we have a value practically equal to this with a depth of moderate thickness, say from 20 to 50 cms. in the case of powerfully excited hydrogen gas.

As I pointed out about twenty-five years ago, it is due to this circumstance that a hydrogen tube appears white or blue-white when viewed end-on and purple when viewed through the side.

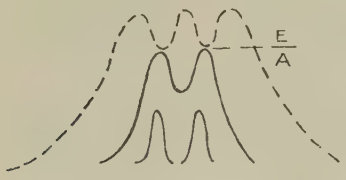
With a layer of fixed thickness as in our vacuum tube we can bring the intensity up to its maximum value (viewed end-on) by merely augmenting the current. It is a little easier, however, to think of the thing in terms of increasing thickness.

As we increase this the intensity of the components of H_α increases, at first rapidly, and then more slowly, finally reaching a maximum value beyond which there is practically no further increase no matter how much we increase the thickness. This intensity $\left(\text{defined by } \frac{E}{A}\right)$ is the intensity of the *dark* lines which we observe when we further increase the thickness, and cause self-reversal. To see how this comes about we have only to assume that the absorption is less, in proportion to the emission for wave-lengths differing slightly from those of the two monochromatic components of H_α, in other words the ratio $\frac{E}{A}$ has a higher value for the *edges* of the lines than for the cores. Representing the intensity increase corresponding to increased thickness by curves in the usual manner, we have the phenomenon of reversal pictured in fig. 2. Curves 1, 1 the two components of H_α for a thin layer, curves 2, 2 for a layer of moderate thickness, showing the cores of the components at their maximum intensity, and curve 3 showing the reversal with dark lines of the same intensity as the bright lines of curve 2.

Previous investigations on the reversal of the hydrogen lines have been made with a source giving a very bright and more or less continuous spectrum, *e. g.* a fine capillary filled with hydrogen at rather high pressure, or a spark under

water, the light being then passed through a tube filled with hydrogen at low pressure and moderately excited. The former arrangement was employed by Ladenburg, and my impression is that certain of his results led him to believe that the ratio $\frac{E}{A}$ varied with the wave-length. This conclusion was questioned by others at the time, but the observations reported in this paper appear to confirm it. The widening of the hydrogen lines with increasing current density is attributed by Merton to a Stark effect resulting from the electric fields of neighbouring atoms, and I am

Fig. 2.



inclined to agree with him that the type of reversal with which we are dealing will have to be explained by taking this effect into account.

The peculiarity of hydrogen seems to lie in the fact that, at least under the conditions described, it is impossible to *reduce* the intensity of the lines by passing the light through a layer of less powerfully excited gas, the latter contributing always more light than it absorbs. No matter how feeble the excitation an actual darkening of the lines seen through the gas can never be detected. This behaviour of the gas is due, I imagine, to the circumstance that we are working with it in the molecular condition. As I showed in an earlier paper, if we operate with dry hydrogen, and free the walls of the tube from adsorbed water vapour, it is possible to excite the tube with a heavy current and obtain only the secondary spectrum, with no trace at all of the Balmer lines.

It may very well turn out that if we have only atomic hydrogen in the absorption and excite it at the lowest possible potential, the conditions for absorbing the light of the Balmer lines may be better realized.

This could be investigated with a long straight tube with three electrode bulbs instead of four as in fig. 1, the central

electrode being connected to one pole of the transformer, and the end electrodes to the other, the one terminating the absorption tube through a very high resistance. A stream of atomic hydrogen will then flow continuously from the powerfully excited tube into the absorption tube, where it is again excited by a lesser potential.

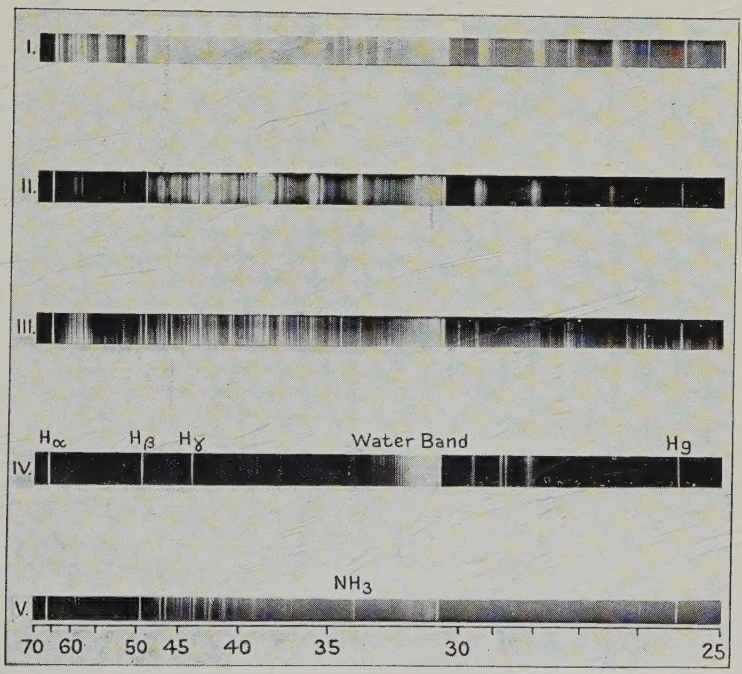
Several years ago, when working with atomic hydrogen I found that, if a small discharge tube was placed in circuit with the tube leading from the main discharge tube to the pump, it could be kept permanently filled with atomic hydrogen, which flowed out of the line of discharge to the pump. If the small tube was operated by a coil giving a very minute spark, the colour of the discharge was white, if no current was passing in the main tube, but purple (Balmer lines only) the moment the main tube was excited. The atomic hydrogen gradually passes back into molecular, owing to catalysis by the walls, as it passes down the tube, so that it is important to have the second tube as close as possible to the main discharge.

The self-reversal described in the earlier part of the paper I have seen only with an echelon, a twenty-plate one by Hilger, with the plates in optical contact. No trace of it appeared with either of the very fine quartz Lummer-Gehrke plates also by Hilger. N. A. Kent and I looked for it in vain in a much larger Lummer plate of glass, but saw it distinctly in his large glass echelon, the plates of which were not in optical contact. The phenomenon resembled so strongly the "grid-structure" shown by echelons of this type, that I was inclined to agree with him at the time, that it might be spurious (*i. e.* instrumental), but Dr. McCurdy and I observed it this winter with another echelon (optical contact) which we borrowed from Princeton, so I feel sure that it is real. Its failure to appear with the Lummer plate is probably due to some peculiarity of the intensity curve of the interference pattern of that instrument.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

SPECTRUM OF SILVER.

I. Arc Lines of Silver. II. Lines of Singly-ionized Silver.



Residual Gas { I. Arc. II. Glow.

Hydrogen ..... { III. Arc showing Balmer Series and Secondary lines. IV. Glow. Water band prominent. V. Glow. Water band disappearing.